Partial Differential Equations and Continuum Mechanics

Proceedings of an International Conference Conducted by the Mathematics Research Center at the University of Wisconsin, Madison, June 7-15, 1960

edited by
RUDOLPH E. LANGER

MADISON · THE UNIVERSITY
OF WISCONSIN PRESS · 1961

GAETANO FICHERA

Linear elliptic equations of higher order in two independent variables and singular integral equations, with applications to anisotropic inhomogeneous elasticity

The purpose of the present paper is to consider a strongly elliptic linear differential equation of order $2m$ in two independent variables and expound a method for obtaining solutions of the Dirichlet problem for such an equation, which are continuous together with derivatives of order m up to the boundary, under rather general conditions on the boundary and the boundary data.

It is well known that in the last ten years much work has been devoted to elliptic equations of higher order by Gårding [6], Viŝik [25], F. John [10], Browder [2], [3], Nirenberg [17], [20], Agmon [1], Hörmander [8], Morrey [17], Miranda [15] and others.

Let us consider the very simple case of the Laplace equations in order to review what methods of approach for this example can be extended to higher order.

We have the energy integral method that many authors have extended to the higher order case with different variants (Gårding, Browder, Nirenberg, etc.). By this method it is proved that a unique solution u of the following integral system

$$\int_A \operatorname{grad} u \cdot \operatorname{grad} v \, d\tau + \int_A v f \, d\tau = 0$$

considered for any v of class C^∞ with support in A, exists in a suitable class of functions. The function u is assumed to be a weak solution of the Dirichlet problem, with homogeneous boundary condition, for the equation $\Delta_2 u = f$.

Under proper assumptions on the given function f, it is proved that u is a classical solution of the problem.

Another classical method for the Laplace equation in two variables consists in representing the solution as a potential of double layer and solving the Dirichlet problem by the classical Fredholm integral equation.

This procedure was extended by Pleijel [22] to the biharmonic equation and by Lopatinsky [13] to more general elliptic equations. But it was Agmon [1] who three years ago succeeded in extending the double layer theory to higher order elliptic equations, in two independent variables, with constant coefficients and no lower order terms.

A third approach, in the case of Laplace equation in two independent variables is by representing the solution as a single layer potential. If we impose Dirichlet boundary condition we obtain an integral equation of the first kind, which does not lead to the information required to solve the problem. However suppose we seek a solution of class C^1 in the closed domain, then by imposing the condition that the tangential derivative on the boundary of the given function be equal to the tangential derivative of the logarithmic potential, we get a singular integral equation, which can be exhaustively treated and whose solution gives the C^1 solution of the Dirichlet problem.

This last approach is the one extended in this paper to a general higher order equation strongly elliptic in two independent variables with variable coefficients and including lower order terms. For simplicity I restrict myself to the case of a single equation with real coefficients. The method, however, applies equally well to general strongly elliptic systems of equations (two independent variables) with complex coefficients.

Moreover I could consider more general boundary conditions, for example, mixed boundary conditions.

The method hinges on two points. Firstly the construction of a proper fundamental solution in the large,of the type which in the 2nd order case is called the principal fundamental solution.

Secondly to establish certain theoretical line-potential results and to write down and discuss the singular integral system that results from the boundary conditions.

§1. Differential equation. Statement of the problem. Parametrix.

We denote by p the ordered pair of nonnegative integers (p_1, p_2) and put $|p| = p_1+p_2$. Let m be a positive integer and the a_{pq} be real functions defined for any $z = x + iy$ of the complex plane and for $0 \leq |p| \leq m$, $0 \leq |q| \leq m$. We shall suppose that $a_{pq} \in C^{2m+\lambda}$ for $|p| + |q| = 2m$. This means that the a_{pq} have continuous partial derivatives of order $2m$ satisfying a uniform Hölder condition in any compact set of the plane. We shall suppose also that $a_{pq} \in C^{\max(|p|,|q|)+\lambda}$ for $|p|+|q| < 2m$. We denote by D^p the partial derivative $\dfrac{\partial^{|p|}}{\partial x^{p_1} \partial y^{p_2}}$ and consider the partial differential operator

$$Eu \equiv D^p a_{pq} D^q u$$

which we write in explicit form as follows

$$Eu = \sum_{k=0}^{2m} a_k(z)\frac{\partial^{2m}u}{\partial x^{2m-k}\partial y^k} + \sum_{r=0}^{2m-1}\sum_{j=0}^{2m-1-r} a_j^{(2m-1-r)}(z)\frac{\partial^{2m-1-r}u}{\partial x^{2m-1-r-j}\partial y^j},$$

where $a_k = \sum_{i+j=k} a_{(m-i,i)(m-j,j)}$. We assume E to be elliptic positive for any z , that is $\sum_{k=0}^{2m} a_k(z)\lambda^{2m-k}\mu^k > 0$ for any non-zero real vector (λ, μ) .

Let A be a bounded domain of the z-plane bounded by a single Jordan curve ∂A , that we suppose to have a uniformly Hölder continuous varying tangent ($\partial A \in C^{1+\lambda}$) . We are concerned with the following Dirichlet problem

D) to find a solution of the differential equation $E(u) = f$ in A satisfying the Dirichlet boundary condition $D^p u = \psi^p$ $(0 \leq |p| \leq m-1)$ continuous together with its derivative of order $\leq m$ in the closure $\bar{A}$ of A and possessing continuous derivatives up to the order $2m$ in A .

f and ψ^p are given functions that must satisfy suitable hypotheses, which we shall specify later.

We put $L(w, z) = \sum_{k=0}^{2m} a_k(z)w^{2m-k}$ and denote by Γ a rectifiable Jordan curve in the complex plane w , lying in the half plane $\mathrm{Im}\, w < 0$ and including all the zeros with negative imaginary part of the polynomial $L(w, z)$. It is evident that when z varies in a compact set we can choose as Γ a curve independent of z .

The following function

$$\mathscr{P}(z, \zeta) = \frac{-1}{2\pi^2(2m-2)!}\,\mathrm{Re}\int_{+\Gamma} \frac{[(x-\xi)w+(y-\eta)]^{2m-2}\log[(x-\xi)w+(y-\eta)]}{L(w,\zeta)}dw$$

is a parametrix in the sense of Hilbert and E. E. Levi for the operator E . For $\log[(x-\xi)w+(y-\eta)]$ we take the principal branch $(-\pi < \arg[(x-\xi)w+(y-\eta)] \leq \pi)$. This, for $x \neq \xi$, is holomorphic in the w-plane cut along the straight half line $\mathrm{Im}\, w = 0$, $\mathrm{Re}\, w \leq -\frac{y-\eta}{x-\xi}$. For $x = \xi$ the logarithm reduces to a constant. Here $\mathscr{P}(x, \zeta)$ is a one-valued function defined for $z \neq \zeta$ and analytic with respect to the variables x, y in any region of the plane not including ζ and of the class $C^{2m+\lambda}$ as a function of ζ in any region not including z . All the derivatives of $\mathscr{P}(z, \zeta)$ can be obtained by differentiating under the integral sign. Since

*

$$D_z^p D_\zeta^q [(x-\xi)w+(y-\eta)]^{2m-2}\log[(x-\xi)w+(y-\eta)] \begin{cases} =O\{[(x-\xi)w+(y-\eta)]^{2m-2|p|-|q|}\log[(x-\xi)w+(y-\eta)]\} \\ \qquad 0 \le |p|+|q| \le 2m-2 \\ =O([(x-\xi)w+(y-\eta)]^{-1}) \quad |p|+|q| = 2m-1 \end{cases}$$

it follows, for z and ζ in any compact set of the plane

$$D_z^p D_\zeta^q \mathscr{P}(z,\zeta) \begin{cases} = O(|z-\zeta|^{2m-2-|p|-|q|}\log|z-\zeta|) & 0 \le |p|+|q| \le 2m-2 \\ = O(|z-\zeta|^{-1}) & |p|+|q| = 2m-1 \,. \end{cases}$$

On the other hand, since $E_\zeta \mathscr{P}(z,\zeta)$ is expressed as a sum of terms such as

$$a_j^{(h)}(\zeta)\frac{-1}{2\pi^2(2m-2)!}\mathrm{Re}\int_{+\Gamma}\{D_\zeta^p[(x-\xi)w+(y-\eta)]^{2m-2}\log[(x-\xi)w+(y-\eta)]\cdot D_\zeta^q[L(w,\zeta)$$

with $0 \le h \le 2m$ $(a_j^{(2m)} \equiv a_j)$, $0 \le |p| < 2m$, $0 \le |q| \le 2m$, it follows that $E_\zeta \mathscr{P}(z,\zeta) = O(|z-\zeta|^{-1})$.

In the usual way the following theorem of the generalized Poisson formula can be proved.

I. <u>Let</u> T <u>be a bounded domain of the</u> z-<u>plane and</u> $\varphi(\zeta)$ <u>a uniform Hölder continuous function in the closure</u> $\overline{T}$. <u>The function</u>

$$u(z) = \int_T \varphi(\zeta)\, \mathscr{P}(\zeta, z)\, d\tau_\zeta$$

<u>belongs to</u> $C^{2m+\lambda}$ <u>for</u> $z \in \overline{T}$ <u>and for</u> $z \in T$ <u>the generalized Poisson formula holds</u>

$$Eu = \varphi(z) + \int_T \varphi(\zeta) E_z \mathscr{P}(\zeta, z)\, d\tau_\zeta \,. \tag{1.1}$$

The same result holds when the roles of z and ζ are interchange in the function $\mathscr{P}(\zeta, z)$.

§2. <u>The principal fundamental solution</u>

The method of Fredholm equations can be applied to second order linear elliptic equations provided a particular fundamental solution can be used which is defined in the whole space, has the proper behavior at infinity and satisfies, as a function of ζ , the adjoint equation. Such a fundamental solution has been constructed by Giraud and we shall call it, following [15], the <u>principal fundamental</u>

solution (p. f. s.). This kind of solution is needed when the method of integral equations, whether of Fredholm or singular type, requires to be extended to higher order equations with variable coefficients§.

However, since we are interested in a bounded domain, we shall construct such p. f. s in a circular domain T containing in its interior $\bar{A}$ and replace the behavior at infinity by proper conditions on the boundary ∂T of T . Let r_1 and r_2 be two positive numbers such that $r_1 < r_2$ and $z \in \bar{A}$ imply $|z| < r_1$. We put $\rho = |z|$, $\Theta = \arg z$ and for any p and q such that $|p| + |q| = 2m$ we consider a function a_{pq} of the class $C^{2m+\lambda}$ satisfying the following conditions

$$\left[\frac{\partial^s a_{pq}}{\partial \rho^s}\right]_{\rho=r_1} = \left[\frac{\partial^s a_{pq}}{\partial \rho^s}\right]_{\rho=r_1}$$

$$\left[\frac{\partial^s a_{pq}}{\partial \rho^s}\right]_{\rho=r_2} \begin{cases} = \dfrac{\partial^s}{\partial \rho^s}\left[\dfrac{1}{i+j+1}\dbinom{m}{\frac{i+j}{2}}\right] & \text{if } i+j \text{ even} \\ = 0 & \text{if } i+j \text{ odd} . \end{cases}$$

$$(p \equiv (m-i, i) \ , \ q \equiv (m-j, j) \ , \ s = 0, 1, \ldots, 2m$$

For $0 < |p|+|q| < 2m$ we consider a function a_{pq} of the class $C^{t+\lambda}$ $(t = \max(|p|, |q|))$ satisfying the conditions

$$\left[\frac{\partial^s a_{pq}}{\partial \rho^s}\right]_{\rho=r_1} = \left[\frac{\partial^s a_{pq}}{\partial \rho^s}\right]_{\rho=r_1} , \quad \left[\frac{\partial^s a_{pq}}{\partial \rho^s}\right]_{\rho=r_2} = 0 \quad (s=0,\ldots,t).$$

Let us put $a_k = \sum_{i+j=k} a_{(m-i,i)(m-j,j)}$ and let r'_1 and r'_2 be two positive numbers such that $r_1 < r'_1 < r'_2 < r_2$ and moreover

$$\sum_{k=0}^{2m} a_k(z)\lambda^{2m-k}\mu^k \geq c(\lambda^2+\mu^2)^m ,$$

(with c a positive constant) for $r_1 \leq |z| \leq r'_1$, $r'_2 \leq |z| \leq r_2$ and for any real vector (λ, μ) .

Let $G(\rho, \tau)$ be the Green's function for the boundary value problem in the interval (r_1, r_2)

$$-\frac{d^{4m+2}v}{d\rho^{4m+2}} = \varphi(\rho) \ , \quad v = \frac{dv}{d\rho} = \cdots = \frac{d^{2m}v}{d\rho^{2m}} = 0 \ \Bigg|_{\rho=r_1}^{\rho=r_2} .$$

Let $p(r'_1, r'_2)$ be a positive number such that for $r'_1 \leq \rho \leq r'_2$

$$v(\rho) = \int_{r_1}^{r_2} G(\rho,\tau)\varphi(\tau)\,d\tau \geq p(r'_1,r'_2) \min_{(r'_1,r'_2)} \varphi(\tau) .$$

We denote by C a positive constant such that $\sum_{k=0}^{2m} a_k(z)\lambda^{2m-k}\mu^k \leq C(\lambda^2+\mu^2)^m$ $(r_1 \leq |z| \leq r_2)$ and assume, for $|p|+|q| = 2m$, a set of arbitrary constant β_{pq} such that

$$\sum_{k=0}^{2m}\left(\sum_{i+j=k} \beta_{(m-i,i)(m-j,j)}\right)\lambda^{2m-k}\mu^k \geq \frac{2C}{p(r'_1,r'_2)}(\lambda^2+\mu^2)^m .$$

Let, for $r_1 \leq |z| \leq r_2$,

$$a^*_{pq}(z) = a_{pq}(z) + \beta_{pq}\int_{r_1}^{r_2} G(\rho,\tau)\,d\tau$$

and consider the functions $(p = (m-i,i)\ ,\ q = (m-j,j))$

$$\widetilde{a}_{pq}\begin{cases} = a_{pq} & |z| \leq r_1 \\ = a^*_{pq} & r_1 \leq |z| \leq r_2 \\ = \begin{cases} = \dfrac{1}{i+j+1}\dbinom{m}{\frac{i+j}{2}} , & \text{if } i+j \text{ even} , \\ = 0 , & \text{if } i+j \text{ odd} . \end{cases} & |z| \geq r_2 \end{cases}$$

and for $0 < |p|+|q| < 2m$

$$\widetilde{a}_{pq}\begin{cases} = a_{pq} & |z| \leq r_1 , \\ = a_{pq} & r_1 \leq |z| \leq r_2 , \\ = 0 & |z| \geq r_2 . \end{cases}$$

Let $\widetilde{E}u$ be the operator $D^p\widetilde{a}_{pq}D^q u$ where $\widetilde{a}_{(0,0)(0,0)} = a_{(0,0)(0,$
It is easily seen that $\widetilde{E}u$ is positive elliptic in any point of the pla
and its coefficients enjoy the same regularity properties as the coefficients of the former operator Eu .

Let $R > r_2$ be a positive number and denote by T the circular domain $|z| < R$. Let u, v be any functions u and v belonging C^{2m} and with support in T . We put

$$\|u\|^2_m = \sum_{0 \leq |p| \leq m} \int_T |D^p u|^2 d\tau\ ,\ B(u,v) = \sum_{|p|,|q|}^{0,m} (-1)^{|p|}\int_T \widetilde{a}_{pq}D^p u$$

According to a fundamental result due to Gårding, two positive constants a_0 and b_0 exist such that, when

$$(-1)^m a_{(0,0)(0,0)}(z) > a_0 \qquad \text{for any } z \in T \tag{2.1}$$

then

$$(-1)^m B(u,u) \geq b_0 \|u\|_m^2 \qquad (\text{support of } u \subset T).$$

Let us suppose (2.1) to be satisfied and assume two positive numbers r_3 and r_4 such that $r_2 < r_3 < r_4 < R$. Let us consider the function defined by

$$a'_{(0,0)(0,0)} \begin{cases} = a_{(0,0)(0,0)} & |z| \leq r_3 \ , \\ = a_{(0,0)(0,0)}\left(1 - \dfrac{\rho - r_3}{r_4 - r_3}\right) & r_3 \leq |z| \leq r_4 \ , \\ = 0 & |z| \geq r_4 \ . \end{cases}$$

Let us denote by E' the operator having as coefficients the functions $\tilde{a}_{pq}$ for $|p|+|q| > 0$ and let $a'_{(0,0)(0,0)}$ be the coefficient of u. If we denote by $B'(u,v)$ the bilinear form corresponding to E', the Gårding inequality

$$(-1)^m B'(u,u) \geq b'_0 \|u\|_m^2 \tag{2.2}$$

$$(b'_0 > 0 \ ; \quad \text{support of } u \subset T)$$

is satisfied since

$$(-1)^m B'(u,u) \begin{cases} = (-1)^m B(u,u) & \text{if } U \subset T_{r_3} \\ = \displaystyle\sum_{k=0}^{m} \binom{m}{k} \int_T \left(\frac{\partial^m u}{\partial x^k \partial y^{m-k}}\right)^2 d\tau + (-1)^m \int_T a'_{(0,0)(0,0)} u^2 dx & \\ & \text{if } U \subset T - T_{r_2} \ . \end{cases}$$

Here U denotes the support of u and T_{r_i} the circular domain $|z| < r_i$. It must be observed that for $|z| \geq r_4$ the operator E' coincides with the iterated Laplace operator $\Delta^{2m} \equiv \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)^m$.

When we shall consider the parametrix $\mathscr{P}(z, \zeta)$ it will be assumed that it refers to the new operator E'. For simplicity of writing, in what follows we shall write the letter E instead of E', the letter a_{pq} instead of $\tilde{a}_{pq}$ for $|p|+|q| > 0$ and $a_{(0,0)(0,0)}$ instead of $a'_{(0,0)(0,0)}$.

Let us denote by $g(z,\zeta)$ the function defined by the following conditions

$$\Delta_z^{2m} g(z,\zeta) = 0 \qquad \text{for } z \in T \ , \ \zeta \in T$$

$$D_z^p g(z,\zeta) = -D_z^p \mathscr{P}(\zeta, z) \qquad 0 \le |p| \le m-1 \ \text{for } z \subset \partial T \ , \ \zeta \in T \ .$$

The existence of such a function is quite classical since it is a regular part of the Green's function for the Lauricella problem (Dirichlet problem) for the polyharmonic equation $\Delta u^{2m} = 0$ (see [19], [21]); $g(z,\zeta)$ is an analytic function of $(x, y\ ;\ \xi, \eta)$, where $z = x+iy$, $\zeta = \xi + i\eta$, in $\overline{T} \times \overline{T}$. Let us consider as a new parametrix in the domain T , for the operator E , the function $\mathscr{S}(z,\zeta) = \mathscr{P}(\zeta, z) + g(z,\zeta)$. For $|z| \ge r_4$, $\mathscr{S}(z,\zeta)$ coincides with the Green's function for the Dirichlet problem in T for the equation $\Delta u^{2m} = 0$. Let $\{\alpha_h(z)\}$ and $\{\beta_h(z)\}$ $(h = 1, \ldots, q)$ be two sets of arbitrary functions belonging respectively to $C^{2m+\lambda}$ and to C^λ and such that the support of any α_h is contained in T . Let μ denote a real parameter. We put

$$u(z) = \int_T \varphi(\zeta)\Big[\mathscr{S}(z,\zeta) + \mu \sum_{h=1}^{q} \alpha_h(z)\beta_h(\zeta)\Big] d\tau_\zeta \ . \qquad (2.3)$$

The function u satisfies the homogeneous boundary condition $D^p u =$ $(0 \le |p| \le m-1)$. If we impose once more the condition $Eu = f$ in any point of T (f a given function, belonging to C^λ), we obtain the Fredholm integral equation

$$f(z) = \varphi(z) + \int_T \varphi(\zeta)\Big[E_z \mathscr{S}(z,\zeta) + \mu \sum_{h=1}^{q} (E\alpha_h(z))\beta_h(\zeta)\Big] d\tau_\zeta \qquad (2.4)$$

whose kernel depends on the parameter μ . We want to prove that a choice of q , α_h , β_h is possible in such a way that for some real μ the associated homogeneous equation

$$0 = \varphi(z) + \int_T \varphi(\zeta)\Big[E_z \mathscr{S}(z,\zeta) + \mu \sum_{h=1}^{q} (E\alpha_h(z))\beta_h(\zeta)\Big] d\tau_\zeta \qquad (2.5)$$

has no eigensolutions. Let us denote by $\mathscr{L}^2$ the Hilbert space of th real functions of integrable square in T with the classical scalar product. Let us consider the operators

$$\mathscr{E}(\varphi) = \int_T \varphi(\zeta) E_z \mathscr{S}(z,\zeta)\, d\tau_\zeta \ , \quad \mathscr{E}^*(\varphi) = \int_T \varphi(\zeta) E_\zeta \mathscr{S}(\zeta, z)\, d\tau_\zeta$$

$$Q(\varphi) = \int_T \varphi(\zeta) \sum_{h=1}^{q} (E\alpha_h(z))\beta_h(\zeta)\, d\tau_\zeta \ .$$

If the equation

$$O = \varphi + \mathscr{E}(\varphi) \tag{2.6}$$

has no eigensolution, then we assume $\mu = 0$. Suppose the equation has an eigensolution. Consider a complete and orthonormal set of linearly independent eigensolutions $\beta_1(z),\dots,\beta_q(z)$ of the equation (2.6). Let $\beta^*_1(z),\dots,\beta^*_q(z)$ be an analogous system for the adjoint equation

$$O = \varphi + \mathscr{E}^*(\varphi) \quad . \tag{2.7}$$

Let us suppose that n is the maximum of the rank of the matrix $\{(E(\alpha_h),\beta^*_k)\}$ $(h,k = 1,\dots,q)$, when $\alpha_1,\dots,\alpha_q$ vary in the class of functions belonging to C^{2m} and with support in T. Suppose $n < q$, and choose $\alpha_1,\dots,\alpha_n$ such that $\det\{(E(\alpha_h),\beta^*_k)\} \neq 0$, $(h,k = 1,\dots,n)$. Let $c_1,\dots,c_{n+1}$ be a non-trivial solution of the homogeneous system $\sum_{k=1}^{n+1} c_k(E(\alpha_h),\beta^*_k) = 0$ $(h = 1,\dots,n)$ and put $\beta^* = \sum_{k=1}^{n+1} c_k\beta^*_k$. The function β^*, since it is an eigensolution of (2.7), belongs to C^{2m}, as is easily seen. Moreover, since $(E(\alpha),\beta^*) = (\alpha, E^*(\beta^*)) = 0$ for any α (with support in T) we have $E^*(\beta^*) = 0$. This implies, by using Green's identity, since $\beta^* = -\mathscr{E}^*(\beta^*)$, that

$$0 = \int_{\partial T} H_\zeta[\mathscr{S}(\zeta,z),\beta^*(\zeta)]\,ds_\zeta \quad , \tag{2.8}$$

where H is a bilinear form operating on the derivatives of $\mathscr{S}$ and those of β^*. On the other hand the right-hand side of (2.8) gives the solution of the Lauricella problem for the equation $\Delta^{2m}u = 0$ such that $D^p u = D^p\beta^*$ on ∂T $(0 \leq |p| \leq m-1)$. This implies that $D^p\beta^* = 0$ $(0 \leq |p| \leq m-1)$ on ∂T. Also from (2.2) it follows that $\beta^* = 0$ in T. This is a contradiction. Thus we have proved $n = q$.

Let us assume the α_h such that $\det\{(E(\alpha_h),\beta^*_k)\} \neq 0$ $(h,k = 1,\dots,q)$. The equation (2.5) can be written

$$0 = \varphi + \mathscr{E}(\varphi) + \mu Q(\varphi) \quad . \tag{2.9}$$

Let $U(\beta^*_1,\dots,\beta^*_q)$ be the linear manifold spanned by all the real linear combinations of functions $\beta^*_1,\dots,\beta^*_q$ and let P be the projector on $V = \mathscr{L}^2 \ominus U(\beta^*_1,\dots,\beta^*_q)$, that is to say on the range of the operator $\mathscr{I} + \mathscr{E}$ ($\mathscr{I} \equiv$ identity operator).

A unique linear bounded operator $\mathscr{R}$ exists such that for any $\psi \in \mathscr{L}^2$ the conditions

$$\mathscr{R}P\psi + \mathscr{E}\mathscr{R}P\psi = P\psi \quad , \qquad (\mathscr{R}P\psi,\beta_h) = 0 \qquad (h = 1,\dots,q)$$

are satisfied. Equation (2.9) implies that

$$\varphi = -\mu \mathscr{R} PQ\varphi + \sum_{h=1}^{q} (\varphi, \beta_h)\beta_h .$$

Let us put $\mathscr{R}PQ = \mathscr{T}$, assume $0 \leq |\mu| < \|\mathscr{T}\|^{-1}$, and consider the operator $\mathscr{S}_\mu = \sum_{k=0}^{\infty} (-\mu)^k \mathscr{T}^k$. Every eigensolution of (2.9) is given by $\varphi = \sum_{h=1}^{q} (\varphi, \beta_h) \mathscr{S}_\mu \beta_h$. Moreover, since $(Q(\varphi), \beta^*_k) = 0$, we ha

$$\sum_{h=1}^{q} (\varphi, \beta_h)(Q \mathscr{S}_\mu \beta_\mu, \beta^*_k) = 0 \quad (k = 1, \ldots, q) ,$$

This implies that $\Delta(\mu) = \det\{Q \mathscr{S}_\mu \beta_h, \beta^*_k)\} = 0 \ (h, k = 1, \ldots, q)$. Since $\Delta(\mu)$ is a holomorphic function of μ in the neighborhood of $\mu = 0$ and

$$\Delta(o) = \det\{(Q\,\beta_h, \beta^*_k)\} = \det\{(E(\alpha_h), \beta^*_k)\} \neq 0$$

it follows that for $\mu \neq 0$ and $|\mu|$ small enough, equation (2.9) has no eigensolution. Let $R(z, \zeta)$ be the resolvent kernel of the Fredho equation (2.4). From (2.3) we deduce

$$u(z) = \int_T f(\zeta) F(z, \zeta)\, d\tau_\zeta$$

where

$$F(z, \zeta) = \mathscr{S}(z, \zeta) + \mu \sum_{h=1}^{q} \alpha_h(z)\beta_h(\zeta) +$$

$$+ \int_T [\, \mathscr{S}(z, w) + \mu \sum_{h=1}^{q} \alpha_h(z)\beta_h(w)]R(w, \zeta)\, d\tau_w .$$

The function $F(z, \zeta)$ is the desired principal fundamental solution. By repeating the argument when considering the adjoint operator E^* instead of E, we get the corresponding function $F^*(z, \zeta)$ and a classical argument proves that $F^*(z, \zeta) = F(\zeta, z)$ (see [15], p. 17) The properties of $F(z, \zeta)$ are obtained by using standard arguments (see [15], p. 51).

What we want to note explicitly are the following properties

$$E_z F(z, \zeta) = 0 \quad (z \in T, \zeta \in T), \quad D_z^p F(z, \zeta) = 0 \quad (z \in \partial T, \ \zeta \in T, \ 0 \leq |p| \leq m-1)$$

$$E^*_\zeta F(z, \zeta) = 0 \quad (z \in T, \ \zeta \in T), \quad D_\zeta^p F(z, \zeta) = 0 \quad (z \in T, \ \zeta \in \partial T, \ 0 \leq |p| \leq m-1)$$

$$D_z^p D_\zeta^q F(z, \zeta) = O(D_z^p D_\zeta^q \ \mathscr{P}(z, \zeta)) \quad (0 \leq |p| + |q| \leq 2m) .$$

In particular

$$\frac{\partial^{2m-1} F(z,\zeta)}{\partial x^{m-h}\partial y^{h}\partial \xi^{m-1-k}\partial \eta^{k}} = (-1)^{m-1} \frac{\partial^{2m-1} \mathscr{P}(z,\zeta)}{\partial x^{2m-1-(h+k)}\partial y^{h+k}} + O(\log|z-\zeta|) \tag{2.10}$$

Theorem I also holds if we replace $\mathscr{P}(\zeta, z)$ by $F(z,\zeta)$. Of course (1.1) must be replaced by the equation $Eu = \varphi$.

Remark: The method outlined for obtaining the p.f.s. works exactly the same in the case of any number of independent variables.

§3. Some results of line-potential theory

Let φ be a real function defined on ∂A and satisfying on ∂A a uniform Hölder condition, briefly $\varphi \in C^{\lambda}(\partial A)$. The following theorem holds

II. With the assumed hypotheses concerning ∂A and φ, for any $z \in A$ let us consider the function

$$v_k(z) = \int_{\partial A} \varphi(\zeta) \frac{\partial^{2m-1}}{\partial x^{2m-1-k}\partial y^{k}} \mathscr{P}(z,\zeta)\, ds_\zeta \qquad (0 \le k \le 2m-1).$$

$v_k(z)$ satisfies at any boundary point z^0 of ∂A the limit relation

$$\lim_{z\to z^0} v_k(z) = \varphi(z^0)\frac{1}{2\pi} \operatorname{Im} \int_{+\Gamma} \frac{w^{2m-1-k}dw}{L(w,z^0)(\dot{x}^0 w + \dot{y}^0)} - \frac{1}{2\pi^2} \operatorname{Re} \int_{\partial A} \varphi(\zeta)\, ds_\zeta \int_{+\Gamma} \frac{w^{2m-1-k}dw}{L(w,\zeta)[(x^0-\xi)w + (y^0-\eta)]}, \tag{3.1}$$

when z tends to z^0 remaining in the interior of A. (The dot denotes differentiation with respect to the arc length on ∂A oriented in the counter-clockwise sense). The integral over ∂A in the right-hand side must be understood as a singular Cauchy integral.

For any fixed w on Γ (Γ chosen independent of z, since z varies in $\bar{A}$) we consider the integral

$$\rho(z,w) = \int_{\partial A} \frac{\varphi(\zeta)\, ds_\zeta}{(x-\xi)w + (y-\eta)} \qquad (z \notin \partial A).$$

By the transformation $z_1 = xw + y$, putting $\zeta_1 = \xi w + \eta$ and denoting by Σ the contour obtained by transforming ∂A, we have

$$\rho(z,w) = \int_{\Sigma} \Phi(\zeta_1)\frac{ds_\zeta}{d\sigma_{\zeta_1}}\,\frac{d\sigma_{\zeta_1}}{z_1-\zeta_1} = \int_{+\Sigma} \Phi(\zeta_1)\frac{ds_\zeta}{d\sigma_{\zeta_1}}\,\frac{d\bar{\zeta}_1}{d\sigma_{\zeta_1}}\,\frac{d\zeta_1}{z_1-\zeta_1}\;.$$

Here σ_{ζ_1} denotes the arc length on Σ (curvilinear abscissa of ζ_1) and $\Phi(\xi w+\eta) \equiv \varphi(\zeta)$. Let z_1^0 be the point on Φ corresponding to the point z^0 on ∂A. We have (for $z \in A$), using the Plemelj formula (see [18], p. 42)

$$\lim_{z\to z^0} \rho(z,w) = \lim_{z_1\to z_1^0}\int_{+\Sigma} \Phi(\zeta_1)\frac{ds_\zeta}{d\sigma_{\zeta_1}}\,\frac{d\bar{\zeta}_1}{d\sigma_{\zeta_1}}\,\frac{d\zeta_1}{z_1-\zeta_1} =$$

$$= -i\pi\,\Phi(z_1^0)\left(\frac{ds_\zeta}{d\sigma_{\zeta_1}}\,\frac{d\bar{\zeta}_1}{d\sigma_{\zeta_1}}\right)_{\zeta_1=z_1^0} + \int_{+\Sigma} \Phi(\zeta_1)\frac{ds_\zeta}{d\sigma_{\zeta_1}}\,\frac{d\bar{\zeta}_1}{d\sigma_{\zeta_1}}\,\frac{d\zeta_1}{z_1^0-\zeta_1} =$$

$$= \frac{-i\pi\varphi(z^0)}{\dot{x}^0 w + \dot{y}^0} + \int_{\partial A} \frac{\varphi(\zeta)\,ds_\zeta}{(x^0-\xi)w+(y^0-\eta)}\;. \tag{3.2}$$

The modulus of continuity of the function

$$\int_{+\Sigma} \Phi(\zeta_1)\frac{ds_\zeta}{d\sigma_{\zeta_1}}\,\frac{d\bar{\zeta}_1}{d\sigma_{\zeta_1}}\,\frac{d\zeta_1}{z_1-\zeta_1}$$

depends on the Hölder coefficient and exponent of $\Phi(z_1)\dfrac{ds_\zeta}{d\sigma_{\zeta_1}}\,\dfrac{d\bar{\zeta}_1}{ds_{\zeta_1}}$, as a function of ζ_1 on Σ, and on a geometrical constant of the curve Σ (see [18], p. 39).

When w varies on Γ all these constants can be chosen independent of w. It follows that the limit relation (3.2) is uniform with respect to w. We have

$$v_k(z) = -\frac{1}{2\pi^2}\,\mathrm{Re}\int_{\partial A}\varphi(\zeta)\,ds_\zeta \int_{+\Gamma} \frac{w^{2m-1-k}}{L(w,\zeta)[(x-\xi)w+(y-\eta)]}\,dw$$

so that

$$\lim_{z\to z^0} v_k(z) = \lim_{z\to z^0}\left\{-\frac{1}{2\pi^2}\,\mathrm{Re}\int_{+\Gamma} w^{2m-1-k}dw\int_{\partial A}\frac{\varphi(\zeta)\,ds_\zeta}{L(w,\zeta)[(x-\xi)w+(y-\eta)]}\right\} =$$

$$= -\frac{\varphi(z^0)}{2\pi}\,\mathrm{Im}\int_{+\Gamma}\frac{w^{2m-1-k}dw}{L(w,z^0)(\dot{x}^0 w+\dot{y}^0)} - \frac{1}{2\pi^2}\,\mathrm{Re}\int_{+\Gamma} w^{2m-1-k}dw\int_{\partial A}\frac{\varphi(\zeta)\,ds_\zeta}{L(w,\zeta)[(x^0-\xi)w-(y^0-\eta)}$$

Since, as it is easily seen, the order of integrations can be changed in the last integral on the right-hand side, (3.1) is proved.

§4. Singular integral system for the Dirichlet problem

Let us now consider problem D) of section 1 and suppose the given real function f to be, uniformly Hölder continuous in $\overline{A}$. Since $u_0(z) = \int_A f(\zeta)F(z,\zeta)d\tau_\zeta$ is a particular solution (of class $C^{2m+\lambda}$ in $\overline{A}$) of the equation $Eu = f$ we have no loss of generality if we consider problem D) for $f \equiv 0$.

Let us assume as given boundary functions $\psi^p \equiv \psi^{(p_1, p_2)}$ continuous real functions satisfying the compatibility conditions

$$\psi^{(p_1,p_2)}(z) - \psi^{(p_1,p_2)}(z_0) = \int_{z_0}^{z} [\psi^{(p_1+1,p_2)}d\xi + \psi^{(p_1,p_2+1)}d\eta] \quad \#$$

$$\int_{\partial A} [\psi^{(p_1+1,p_2)}dx + \psi^{(p_1,p_2+1)}dy] = 0 \tag{4.1}$$

$$(0 \leq p_1 \leq m-2\ ,\ 0 \leq p_2 \leq m-2)\ .$$

Moreover since we seek a solution of class C^m in $\overline{A}$ we must suppose that ψ^p for $|p| = m-1$ possesses a continuous derivative with respect to s . Since the mere continuity of $\frac{\partial \psi^p}{\partial s}$ $(|p| = m-1)$ does not insure that the solution belongs to C^m— as classical examples in the case $m = 1$ show—we shall suppose that $\frac{\partial \psi^p}{\partial s}$ is uniformly Hölder continuous on ∂A .

Put $p \equiv (m-1-h, h)$ and put $\psi_h = \frac{\partial \psi^p}{\partial s}$. Let $\varphi_k(\zeta)$ $(k = 0, \ldots, m-1)$ be real functions belonging to $C^\lambda(\partial A)$ and consider the function

$$u(z) = \sum_{k=0}^{m-1} \int_{\partial A} \varphi_k(\zeta) \frac{\partial^{m-1}}{\partial \xi^{m-1-k} \partial \eta^k} F(z,\zeta)\, ds_\zeta\ . \tag{4.2}$$

$u(z)$ is of class C^m in $\overline{A}$ and C^{2m} in A and satisfies in A the equation $Eu = 0$. We want to determine the φ_k in such a way that u satisfies the boundary conditions

$$\frac{\partial}{\partial s} u_{x^{m-1-h} y^h} = \psi_h\ . \tag{4.3}$$

From (2.10) and (3.1) we obtain for any $z \in \partial A$

$$(-1)^{m-1}\psi_h(z) = \sum_{k=0}^{m-1} \left\{ \frac{-\varphi_k(z)}{2\pi} \operatorname{Im} \int_{+\Gamma} \frac{w^{2m-2-(h+k)}}{L(w,z)}\, dw \right. -$$

$$\left. - \frac{1}{2\pi^2} \operatorname{Re} \int_{\partial A} \varphi_k(\zeta)\, ds_\zeta \int_{+\Gamma} \frac{w^{2m-2-(h+k)} (\dot{x}w+\dot{y})}{L(w,\zeta)[(x-\xi)w+(y-\eta)]} dw + \int_{\partial A} \varphi_k(\zeta)\, M_{hk}^{(1)}(z,\zeta)\, ds_\zeta \right\} . \tag{4.4}$$

The kernels $M_{hk}^{(1)}(z,\zeta)$ are regular for $z \neq \zeta$ and are $O(\log|z-\zeta|)$.

Let us introduce the function defined for $w \subset \Gamma$, $(z,\zeta) \in \partial A \times \partial A$

$$K(w,z,\zeta) \begin{cases} = \dfrac{\dot{\bar{\zeta}}(\dot{x}w+\dot{y})}{\dfrac{x-\xi}{z-\zeta}w + \dfrac{y-\eta}{z-\zeta}} & , \quad z \neq \zeta \\ = 1 & , \quad z = \zeta \end{cases}$$

for any $w \in \Gamma$, K is uniformly Hölder continuous with respect to (z,ζ) in $\partial A \times \partial A$ (see [4], p. 155).

Put $H(w,z,\zeta) = K(w,z,\zeta) - 1$. We have

$$\frac{\dot{\bar{\zeta}}(\dot{x}w+\dot{y})}{(x-\xi)w+(y-\eta)} = \frac{1}{z-\zeta} + \frac{H(w,z,\zeta)}{z-\zeta}.$$

Then

$$\int_{\partial A} \varphi_k(\zeta)\, ds_\zeta \int_{+\Gamma} \frac{w^{2m-2-(h+k)}(\dot{x}w+\dot{y})}{L(w,\zeta)[(x-\xi)w+(y-\eta)]}\, dw =$$

$$= \int_{+\Gamma} \varphi_k(\zeta) d\zeta \int_{+\Gamma} \frac{w^{2m-2-(h+k)}}{L(w,z)} \frac{dw}{z-\zeta} + \int_{+\partial A} \varphi_k(\zeta) d\zeta \int_{+\Gamma} \frac{H_1(w,z,\zeta)}{z-\zeta} w^{2m-2-(h+k)} dw,$$

where

$$H_1(w,z,\zeta) = \frac{H(w,z,\zeta)}{L(w,\zeta)} + \frac{1}{L(w,\zeta)} - \frac{1}{L(w,z)}.$$

On the other hand we have

$$\mathrm{Re} \int_{+\Gamma} \frac{w^{2m-2-(h+k)}}{L(w,z)} dw \int_{+\partial A} \frac{\varphi_k(\zeta)\, d\zeta}{z-\zeta} =$$

$$= -\left(\mathrm{Re}\int_{+\Gamma} \frac{w^{2m-2-(h+k)}}{L(w,z)} dw\right) \int_{+\partial A} \frac{\varphi_k(\zeta)\, d\zeta}{\zeta - z} +$$

$$- i \int_{+\Gamma} \overline{\frac{w^{2m-2-(h+k)}}{L(w,z)} dw} \int_{+\partial A} \varphi_k(\zeta) \frac{\partial}{\partial n_\zeta} \log|z-\zeta|\, ds_\zeta$$

(n_ζ is the inward normal at ∂A in ζ).

We can now write the singular system (4.4) in the canonical form (see [18], p. 416).

$$(-1)^{m-1} 2\pi \psi_h(z) = \sum_{k=0}^{m-1} \left\{ a_{hk}(z)\varphi_k(z) + \frac{b_{hk}(z)}{\pi i} \int_{+\partial A} \frac{\varphi_k(\zeta)}{\zeta - z} d\zeta + \right.$$

$$\left. + \int_{+\partial A} \varphi_k(\zeta) M_{hk}(z,\zeta) d\zeta \right\} , \quad (h = 0, \dots, m-1) , \tag{4.5}$$

where

$$a_{hk}(z) = - \operatorname{Im} \int_{+\Gamma} \frac{w^{2m-2-(h+k)}}{L(w,z)} dw \ , \ b_{hk}(z) = i \operatorname{Re} \int_{+\Gamma} \frac{w^{2m-2-(h+k)}}{L(w,z)} dw$$

$$M_{hk}(z,\zeta) = O\left(\frac{1}{|z-\zeta|^{1-\lambda}}\right) , \qquad (0 < \lambda \leq 1) .$$

Let us denote by $\mathscr{A}$ and $\mathscr{B}$ the matrices $\{a_{hk}\}$, $\{b_{hk}\}$, $(h, k = 0, \dots, m-1)$. In order to show that the system (4.5) is of regular type ([18], p. 417) we must prove that $\delta(z) = \det\{\mathscr{A} - \mathscr{B}\}\det\{\mathscr{A} + \mathscr{B}\}$ never vanishes on ∂A. This amounts to proving that

$$\delta_0(z) \equiv \det \left\{ \int_{+\Gamma} \frac{w^{2m-2-(h+k)}}{L(w,z)} dw \right\} \neq 0 \qquad (z \in \partial A) .$$

Suppose, for some z, $\delta(z) = 0$. Let $c_0, \dots, c_{m-1}$ be a non-trivial solution of the system

$$\sum_{k=0}^{m-1} c_k \int_{+\Gamma} \frac{w^{2m-2-(h+k)}}{L(w,z)} dw = 0 \qquad (h = 0, \dots, m-1) .$$

If we put $P(w) = \sum_{k=0}^{m-1} c_k w^{m-1-k}$, for any polynomial $Q(w)$ of degree $\leq m-1$, the following equation holds

$$\int_{+\Gamma} \frac{Q(w)P(w)}{L(w,z)} dw = 0 . \tag{4.6}$$

We can choose $Q(w)$ in such a way that the function $Q(w)P(w)[L(w,z)]^{-1}$ has a simple pole and no other singularity in the domain bounded by Γ. This contradicts (4.6).

Deforming Γ into the real axis, we get

$$\int_{+\Gamma} \frac{w^{2m-2-(h+k)}}{L(w,z)} dw = \int_{-\infty}^{+\infty} \frac{u^{2m-2-(h+k)}}{L(u,z)} du$$

and, therefore, $a_{hk}(z) \equiv 0$. From this we deduce that, denoting by κ the index ([18], p. 419) of the system (4.5) we have $\kappa = 0$. This means that for the system (4.5) the same Fredholm theorems of the regular integral system hold.

§5. Determination of the eigensolutions of the homogeneous integral system. Existence theorem

Let us consider the associated homogeneous system

$$0 = \sum_{k=0}^{m-1} \left\{ \frac{b_{hk}(z)}{\pi i} \int_{+\partial A} \frac{\varphi_k(\zeta)}{\zeta - z} d\zeta + \int_{+\partial A} \varphi_k(\zeta) M_{hk}(z,\zeta)\, d\zeta \right\} \qquad (5.1)$$
$$(h = 0, \dots, m-1) \;.$$

We shall say that the $(m-1)$-vector $g(z) \equiv (g_0(z), \dots, g_{m-1}(z))$ is an eigensolution of (5.1) of the first kind when some of the $g_k(z)$ are not identically vanishing in ∂A and

$$\sum_{k=0}^{m-1} \int_{\partial A} g_k(\zeta) \frac{\partial^{m-1}}{\partial \xi^{m-1-k} \partial \eta^k} F(z,\zeta)\, ds_\zeta = 0 \qquad (5.2)$$

for any $z \in T$. An eigensolution of (5.1) that is not of the first kind will be called of the second kind. Since

$$\sum_{j=0}^{r-1} \binom{r-1}{j} \int_{\partial A} \frac{\partial^{m-1} F(z,\zeta)}{\partial \xi^{m-1-k-j} \partial \eta^{k+j}} d(\xi^{r-1-j} \eta^j) \equiv 0$$
$$(r = 2, \dots, m \;;\; k = 0, \dots, m-r)$$

for any $z \in T$, we obtain $\frac{m(m-1)}{2}$ eigensolutions $g^{(r,k)}$ of the first kind by defining the components of $g^{(r,k)}$ as follows

$$g_0^{(r,k)} = 0, \dots, g_{k-1}^{(r,k)} = 0 \;,\; g_k^{(r,k)} = \binom{r-1}{0} \frac{d\xi^{r-1}}{ds}, \dots, g_{k+j}^{(r,k)} = \binom{r-1}{j} \frac{d\xi^{r-1-}}{ds}$$
$$\dots, g_{k+r-1}^{(r,k)} = \binom{r-1}{z-1} \frac{d\eta^{r-1}}{ds} \;,\; g_{k+r}^{(r,k)} = 0, \dots, g_{m-1}^{(r,k)} = 0 \;.$$

It is easily seen that the $g^{(r,k)}$ are linearly independent.

We prove now that any eigensolution of the first kind is a linear combination with constant coefficients of the $g^{(r,k)}$. Let $g \equiv (g_1, \dots, g_{m-1})$ be an eigensolution of the first kind. Since any function $f(\zeta)$ of class C^∞ and support in T can be represented, for $\zeta \in T$, as follows

$$f(\zeta) = \int_T \mu(z) F(z,\zeta)\, d\tau_z \qquad (\mu = E^*(f)) \;,$$

from (5.2) it follows

$$\sum_{k=0}^{m-1} \int_{\partial A} g_k(\zeta) \frac{\partial^{m-1} f}{\partial \xi^{m-1-k} \partial \eta^k} ds_\zeta = 0 \ .$$

The statement regarding g is a consequence of the arbitrariness of the function f.

The dimension of the linear manifold $\mathscr{M}$ of the eigensolutions of (5.1) is equal to the number of linearly independent conditions of compatibility that must be satisfied by the known terms ψ_h of system (4.5). This is a consequence of the fact that $\kappa = 0$.

Since the ψ_h are related to the function $u(z)$ defined by (4.2), by means of the equations (4.3) we obtain the following independent necessary conditions for the solvability of system (4.5).

$$\int_{\partial A} \psi_h = 0 \qquad \sum_{j=0}^{r-1} \binom{r-1}{j} \int_{\partial A} x^{r-i-j} y^j \psi_{k+j} ds = 0 \quad (5.3)$$

$$(h = 0, 1, \ldots, m-1) \qquad (r = 2, \ldots, m \ ; \ k = 0, \ldots, m-r) \ .$$

It follows that $d = \dim \mathscr{M} \geq \frac{m(m+1)}{2}$.

We prove now that $d \leq m^2$. Let us suppose $d > m^2$, then we have a complete system in $\mathscr{M}$ constituted by $\frac{m(m-1)}{2}$ eigensolutions of the first kind and $\frac{m(m+1)}{2} + q$ $(q > 0)$ eigensolutions of the second kind.

If $b \equiv (b_0, \ldots, b_{m-1})$ is an eigensolution of the second kind, it is on ∂A

$$v(z) = \sum_{k=0}^{m-1} \int_{\partial A} b_k(\zeta) \frac{\partial^{m-1}}{\partial \xi^{m-1-k} \partial \eta^k} F(z, \zeta) ds_\zeta = \sum_{i,j}^{0, m-1} c_{ij} x^i y^j \ .$$

Since $q > 0$, it is possible to choose a linear combination of the eigensolutions of the second kind, with some coefficients different from zero, in such a way that the corresponding eigensolution is of the first kind. This is a contradiction.

Let us assume $d = \frac{m(m-1)}{2} + r$ $\left(m \leq r \leq \frac{m(m+1)}{2}\right)$. Let us now consider the adjoint homogeneous system, which we write as follows

$$0 = \sum_{h=0}^{m-1} \int_{\partial A} \nu_h(z) \frac{\partial}{\partial s_z} \frac{\partial^{2m-2} F(z, \zeta)}{\partial x^{m-h-1} \partial y^h \partial \xi^{m-k-1} \partial \eta^k} ds_z \ . \qquad (5.4)$$

$$(k = 0, \ldots, m-1)$$

We shall call eigensolution of the first kind for the system (5.4) any eigensolution $\gamma \equiv (\gamma_0, \ldots, \gamma_{m-1})$ such that

$$\sum_{h=0}^{m-1} \int_{\partial A} \gamma_h(z) \frac{\partial}{\partial s_z} \frac{\partial^{m-1} F(z,\zeta)}{\partial x^{m-1-h} \partial y^h} ds_z = 0$$

for any $\zeta \in T$.

A similar argument, as in the case of system (5.1), proves that $\frac{m(m+1)}{2}$ linearly independent eigensolutions of the first kind exist and any eigensolution of the first kind is expressed as a linear combination of these $\frac{m(m+1)}{2}$ eigensolutions. An eigensolution of (5.4), which is not of the first kind, is called of the second kind. A complete system in the linear manifold of the eigensolutions of (5.4) is constituted by $\frac{m(m+1)}{2}$ eigensolutions of the first kind and $r-m$ of the second kind. Let $\nu^{(1)}, \ldots, \nu^{(r-m)}$ be the eigensolutions of the second kind. If $r > m$ it is possible to choose $r-m$ points $\zeta_1, \ldots, \zeta_{r-m}$ in $T - \overline{A}$ such that

$$\det\left\{\sum_{h=0}^{m-1} \int_{\partial A} \nu_h^{(i)}(z) \frac{\partial}{\partial s_z} \frac{\partial^{m-1} F(z,\zeta_k)}{\partial x^{m-1-h} \partial y^h} ds_z\right\} \neq 0 .$$

$$(i, k = 1, \ldots, r-m)$$

Let us suppose that this is not true and be $n < r-m$ the rank of the corresponding matrix. Let $\zeta_1, \ldots, \zeta_n$ be n points in $T - \overline{A}$ such that

$$\det\left\{\sum_{h=0}^{m-1} \int_{\partial A} \nu_h^{(i)}(z) \frac{\partial}{\partial s_z} \frac{\partial^{m-1} F(z,\zeta_k)}{\partial x^{m-1-h} \partial y^h} ds_z\right\} \neq 0 .$$

$$(i, k = 1, \ldots, n)$$

Let $c_1, \ldots, c_{n+1}$ a non-trivial solution of the homogeneous system

$$\sum_{i=1}^{n+1} c_i \sum_{h=0}^{m-1} \int_{\partial A} \nu_h^{(i)}(z) \frac{\partial}{\partial s_z} \frac{\partial^{m-1} F(z,\zeta_k)}{\partial x^{m-1-h} \partial y^h} ds_z = 0 .$$

$$(k = 1, \ldots, n)$$

Let us assume $\nu = \sum_{i=1}^{n+1} c_i \nu^{(i)}$. For any $\zeta \in T - \overline{A}$ it is

$$v_0(\zeta) \equiv \sum_{h=0}^{m-1} \int_{\partial A} \nu_h(z) \frac{\partial}{\partial s_z} \frac{\partial^{m-1} F(z,\zeta)}{\partial x^{m-1-h} \partial y^h} ds_z = 0 .$$

We suppose now $\partial A \in C^{1+\lambda}$ with $\lambda > \frac{1}{2}$. Then it is $\nu_h \in C^{\lambda}(\partial A)$ with $\lambda > \frac{1}{2}$ and, consequently, $\|v_0\|_m < +\infty$ (see [18], p. 51). It follows $v_0(\zeta) \equiv 0$ in T. Then ν is of the first kind. This is impossible.

Let us put for $z \in \partial A$ $\quad f_h^{(k)} = \frac{\partial}{\partial s} \frac{\partial^{m-1} F(z, \zeta_k)}{\partial x^{m-1-h} \partial y^h}$. It is possible to determine the constants $a_1, \ldots, a_{r-m}$ in such a way that $\widetilde{\psi}_h = \psi_h - \sum_{k=1}^{r-m} a_k f_h^{(k)}$ verify the condition

$$\sum_{h=0}^{m-1} \int_{\partial A} \widetilde{\psi}_h \nu_h ds \equiv (\widetilde{\psi}, \nu) = 0 \qquad \pm$$

for any eigensolution of (5.4). The constants a_k are expressed as follows

$$a_k = (\psi, \sum_{i=1}^{r-m} A_{ik} \nu^{(i)}) \equiv (\psi, \mu^{(k)})$$

where A_{ik} is a known constant.

The integral system (4.5) admits one solution when $r = m$. In the case $r \geq m$, one solution exists if ψ_h $(h = 0, \ldots, m-1)$ is replaced by $\widetilde{\psi}_h$.

Let $b^{(1)}, b^{(2)}, \ldots, b^{(r)}$ be r eigensolutions of the second kind for the system (5.1) such that, together with the $\frac{m(m-1)}{2}$ eigensolutions $g^{(r,k)}$ they constitute a complete set in $\mathscr{M}$.

For every m-vector $\varphi \equiv (\varphi_0, \ldots, \varphi_{m-1})$ we put, for $z \in T$,

$$\mathscr{F}(\varphi) = \sum_{h=0}^{m-1} \int_{\partial A} \varphi_h(\zeta) \frac{\partial^{m-1} F(z, \zeta)}{\partial \xi^{m-1-h} \partial \eta^h} ds_\zeta \; .$$

Let us consider the functions $v^{(k)}(z) = \mathscr{F}(b^{(k)})$, $k = 1, \ldots, r$; $v^{(k)}$ is a solution of $Ev^{(k)} = 0$ in A and in $T - \overline{A}$, of class $C^{m+\lambda}$ in $\overline{A}$ and $\overline{T - \overline{A}}$ and verifies the boundary conditions $\frac{\partial}{\partial s} \frac{\partial^{m-1} v}{\partial x^{m-1-h} \partial y^h} = 0$, $(h = 0, \ldots, m-1)$ on ∂A and $D^p v = 0$, $(0 \leq |p| \leq m-1)$ on ∂T .

Let f and g be two real functions defined on ∂A . We consider on ∂A the scalar product

$$\int_{\partial A} f g \, ds = [f, g] \; .$$

Let us suppose that the $\sigma = \frac{m(m+1)}{2}$ functions $x^i y^j$ $(i, j = 0, \ldots, m-1)$ are linearly independent on ∂A . This means that the curve ∂A is not algebraic. We have no loss of generality in assuming this hypothesis since we can always use a proper change of coordinates such that ∂A is transformed into a non-algebraic curve.

With respect to the introduced scalar product we orthonormalize the system of σ functions $x^i y^j$ $(i, j = 0, \ldots, m-1)$. Let $h^{(1)}, \ldots,$ l be the corresponding orthonormal system. We have on ∂A

$$v^{(k)}(z) = \sum_{i=1}^{\sigma} [v^{(k)}, h^{(i)}] h^{(i)}(z) .$$

The rank q of the matrix $\{[v^{(k)}, h^{(i)}]\}$ $(k = 1, \ldots, r ; \quad i = 1, \ldots,$ is r , since for $q < r$ it would be possible to choose r constants $c_1, \ldots, c_r$ such that $\sum_{k=1}^{r} c_k b^{(k)}$ be an eigensolution of the first kind.

Let $u(z)$ be a solution of the equation $Eu = 0$ of class $C^{m+\lambda}$ in $\overline{A}$, that admits for $z \in \overline{A}$ the representation

$$u(z) = \mathscr{F}(\varphi) + \sum_{n=1}^{r-m} a_n F(z, \zeta_n) + \sum_{k=1}^{r} c_k v^{(k)} ; \qquad (5.5)$$

Then φ is a solution of (4.5) where ψ_h is replaced by $\widetilde{\psi}_h$. We suppose that φ is a solution satisfying the conditions $(\varphi, b^{(k)}) = 0$, $(k = 1, \ldots, r)$. The vector φ is expressed by $\varphi = \mathscr{R}\widetilde{\psi}$ where $\mathscr{R}$ is assumed to be a well-determined linear transformation (resolvent transformation) operating on $\widetilde{\psi}$. The properties of $\mathscr{R}$ are well known from the theory of singular integral equations. We have

$$[\mathscr{F}\mathscr{R}\widetilde{\psi}, h^{(i)}] = (\widetilde{\psi}, \mathscr{R}^* \mathscr{F}^* h^{(i)})$$

with a self-explanatory meaning for the transformations $\mathscr{F}^*$ and $\mathscr{R}^*$. From (5.5) it follows

$$[u, h^{(i)}] - (\psi, \omega^{(i)}) = \sum_{k=1}^{r} c_k [v^{(k)}, h^{(i)}] , \qquad (5.6)$$

where

$$\omega^{(i)} = \mathscr{R}^* \mathscr{F}^* h^{(i)} - \sum_{k=1}^{r-m} (f^{(k)}, \mathscr{R}^* \mathscr{F}^* h^{(i)}) \mu^{(k)} + \sum_{k=1}^{r-m} [F(z, \zeta_k), h^{(i)}] \mu$$

Let us suppose $\Delta_0 = \det\{[v^{(k)}, h^{(i)}]\} \neq 0$, $(k, i = 1, \ldots, r)$. From (5.6) it follows for $\sigma > r$

$$[u, \Delta_0 h^{(r+j)} + \sum_{k=1}^{r} B_{jk} h^{(k)}] - (\psi, \Delta_0 \omega^{(r+j)} + \sum_{k=1}^{r} B_{jk} \omega^{(k)}) = 0$$
$$(j = 1, \ldots, \sigma - r) . \qquad (5.7)$$

The B_{jk} are constants independent of u . For $\sigma > r$ there must exist some $\zeta^* \in T - \overline{A}$ such that $u = F(z, \zeta^*)$ does not admit the representation (5.5). In the opposite case we must have

$$[F(z,\zeta),\Delta_0 h^{(r+j)}(z)+\sum_{k=1}^{r} B_{jk} h^{(k)}(z)] - (\Phi(z,\zeta),\Delta_0\omega^{(r+j)}(z)+\sum_{k=1}^{r} B_{jk}\omega^{(k)}(z)) = 0 \qquad (5.8)$$

for any $\zeta \in T-\overline{A}$. $\Phi(z,\zeta)$ denotes the vector $\psi_{(i)}$ corresponding to $F(z,\zeta)$ as a function of z . Since the vectors $\omega^{(i)}$ are uniformly Hölder-continuous on ∂A with an exponent $\lambda > \frac{1}{2}$, (5.8) holds for any $\zeta \in T$. It follows that (5.7) must be satisfied for any u of class C^∞ with support in T . This implies

$\Delta_0 h^{(r+j)} + \sum_{k=1}^{r} B_{jk} h^{(k)} \equiv 0$ on ∂A . This is a contradiction. We put

$$v^{(r+1)}(z) = -F(z,\zeta^*) + \sum_{n=1}^{r-m} (\Phi(z,\zeta^*),\mu^{(n)}(z))F(z,\zeta_n) + \mathscr{F}(\varphi)$$

where $\varphi(z) = \mathscr{R}[\Phi(z,\zeta^*) - \sum_{n=1}^{r-m} (\Phi(z,\zeta^*),\mu^{(n)}(z))f^{(n)}(z)]$. The function $v^{(r+1)}$ is a solution of class $C^{m+\lambda}$ in $\overline{A}$ and in $\overline{A-\overline{T}}$ of the equation $Eu = 0$ which satisfies the same boundary conditions as $v^{(k)}$ $(k = 1,\dots,r)$. The rank q of the matrix $\{[v^{(k)},h^{(i)}]\}$, $(k=1,\dots,r+1\ ;\ i=1,\dots,\sigma)$ is $r+1$. Let us suppose $q < r+1$. Then it is possible to find r constants such that $v^{(r+1)} = \sum_{k=1}^{r} a_k v^{(k)}$ in $\overline{A}$. This implies that $F(z,\zeta^*)$ admits the representation (5.5).

By iterating the procedure we determine $\sigma-r$ functions $v^{(r+1)},\dots,v^{(\sigma)}$ such that

$$\det\{[v^{(k)},h^{(i)}]\} \neq 0 \qquad (k,i = 1,\dots,\sigma) . \qquad (5.9)$$

For proving the existence theorem we first solve the system (4.5) where ψ_h is replaced by $\widetilde{\psi}_h$ (for $r > m$) . Thereafter we determine the constants $c_1,\dots,c_\sigma$ in such a way that the function

$$u(z) = \mathscr{F}\mathscr{R}\widetilde{\psi} + \sum_{n=1}^{r-m} (\psi,\mu^{(n)})F(z,\zeta_n) + \sum_{k=1}^{\sigma} c_k v^{(k)}$$

satisfies the boundary condition $D^p u = \psi^p$ $(0 \le |p| \le m-1)$. This is possible since (4.1),(5.9). Thus we have proved the following existence theorem

III. <u>In the assumed hypotheses for the operator</u> E , <u>the domain</u> A , <u>the known term</u> f <u>and the boundary function</u> ψ^p $(0 \le |p| \le m-1)$ <u>one (and only one) solution of the problem D) exists of class</u> C^{2m} <u>in</u> A <u>and</u> C^m <u>in</u> $\overline{A}$.

§6. The equation $Eu + \lambda u = f$.

We want now to consider problem D) for the equation $Eu+\lambda u = f$, where λ is a real constant. We denote this problem by D_λ and assume on f and ψ^p the same hypotheses as in the considered case $\lambda = 0$.

Let $G(z, \zeta)$ be the Green's function for problem D) with respect to the operator E . We have $G(z, \zeta) = g(z, \zeta) + F(z, \zeta)$ where $g(z, \zeta)$ is defined by the conditions

$$E_z g(z, \zeta) = 0 \qquad (z, \zeta) \in A ,$$

$$D_z^p g(z, \zeta) = - D_z^p F(z, \zeta) \qquad (0 \leq |p| \leq m-1 , \ z \in \partial A , \ \zeta \in A).$$

Let u_0 be the function determined by the conditions

$$Eu_0 = 0 \ \text{ in } A \qquad D^p u_0 = \psi^p \ \text{ on } \partial A \qquad (0 \leq |p| \leq m-1) .$$

A solution u of the problem considered exists when and only when a solution ρ of the following Fredholm integral equation in A exists

$$\rho(z) + \lambda \int_A G(z, \zeta) \rho(\zeta) = f(z) - \lambda u_0(z) . \qquad (6.1)$$

Then we have

$$u(z) = u_0(z) + \int_A \rho(\zeta) G(z, \zeta) d\tau_\zeta .$$

It follows that if λ is not an eigenvalue for (6.1), we have one and only one solution of problem D_λ . If λ is an eigenvalue we have a linear manifold of finite dimension of eigensolutions η for the problem D_λ^* adjoint to D_λ . A solution of problem D_λ exists when and only when the following conditions are satisfied

$$\int_A (f(z) - \lambda u_0(z)) \eta(z) d\tau = 0 \qquad (6.2)$$

for every eigensolution η .

Suppose η be of class C^{2m-1} in $\overline{A}$, then it is possible to prove that

$$-\lambda \int_A u_0 \eta d\tau = \int_{\partial A} \mathscr{K}(\psi^p, D^p \eta) ds$$

where $\mathscr{K}$ is a bilinear form operating on $D^p u_0$, $(0 \leq |p| \leq m-1)$ and $D^q \eta$, $(m \leq |q| \leq 2m-1)$.

The compatibility conditions are the following

$$\int_A f \eta d\tau + \int_{\partial A} \mathscr{K}(\psi^p, D^p \eta) ds = 0 . \qquad (6.3)$$

IV. For the problem D_λ the alternative theorem holds. There exists a finite or countable (or empty) set of eigenvalues. When the eigensolutions of D_λ^* are of class C^{2m-1} in A, the compatibility conditions can be written as (6. 3).

§7. Application to anisotropic inhomogeneous plane elastic systems

$$
\begin{aligned}
e_{xx} &= \ell_{11}\widehat{xx} + \ell_{12}\widehat{yy} + \ell_{13}\widehat{xy} \ , \\
e_{yy} &= \ell_{12}\widehat{xx} + \ell_{22}\widehat{yy} + \ell_{23}\widehat{xy} \ , \\
2e_{xy} &= \ell_{13}\widehat{xx} + \ell_{23}\widehat{yy} + \ell_{33}\widehat{xy} \ ,
\end{aligned}
\tag{7. 1}
$$

where e_{xx}, e_{xy}, e_{yy} are the components of the plane deformation and $\widehat{xx}$, $\widehat{xy}$, $\widehat{yy}$ the stress components**. In the general case of an anisotropic inhomogeneous system, the ℓ_{ij} are arbitrary functions of z, only subjected to some regularity conditions (continuity, differentiability, etc.) and such that the quadratic form $\ell_{ij}(z)\lambda_i\lambda_j$ in the real variables λ_1, λ_2, λ_3 be positive definite. We have for the strain components the compatibility condition

$$
\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y}
\tag{7. 2}
$$

and for the stresses components the indefinite equation of equilibrium

$$
\begin{aligned}
\frac{\partial\widehat{xx}}{\partial x} + \frac{\partial\widehat{xy}}{\partial y} &= b_1 \\
\frac{\partial\widehat{xy}}{\partial x} + \frac{\partial\widehat{yy}}{\partial y} &= b_2
\end{aligned}
\tag{7. 3}
$$

where $b \equiv (b_1, b_2)$ is a given vector (body-vector).

Let us suppose that a particular solution of (7. 3) is given: $\widehat{xx}_0$, $\widehat{xy}_0$, $\widehat{yy}_0$ and that the equilibrium problem is studied in the simply connected domain A considered in the previous sections. Then by introducing the Airy's stress function u, we have

$$
\begin{aligned}
\widehat{xx} = \widehat{xx}_0 + \frac{\partial^2 u}{\partial y^2} \ , \qquad \widehat{yy} = \widehat{yy}_0 + \frac{\partial^2 u}{\partial x^2} \\
\widehat{xy} = \widehat{xy}_0 - \frac{\partial^2 u}{\partial x \partial y} \ .
\end{aligned}
\tag{7. 4}
$$

By substituting the expressions of e_{xx}, e_{yy}, e_{xy} given by (7. 1) in (7. 2) we get

$$\frac{\partial^2}{\partial x^2}(\ell_{12}\widehat{xx} + \ell_{22}\widehat{yy} + \ell_{23}\widehat{xy}) -$$
$$-\frac{\partial^2}{\partial x\partial y}(\ell_{13}\widehat{xx} + \ell_{23}\widehat{yy} + \ell_{33}\widehat{xy}) + \tag{7.5}$$
$$+\frac{\partial^2}{\partial y^2}(\ell_{11}\widehat{xx} + \ell_{12}\widehat{yy} + \ell_{13}\widehat{xy}) = 0 .$$

Let us now substitute in (7.5) the stresses as given by (7.4) and denote by f the result of replacing $\widehat{xx}$, $\widehat{yy}$, $\widehat{xy}$, by $\widehat{xx}_0$, $\widehat{yy}_0$, $\widehat{xy}_0$ in the left hand side of (7.5). We get

$$E(u) \equiv \frac{\partial^2}{\partial x^2}\left(\ell_{12}\frac{\partial^2 u}{\partial y^2} + \ell_{22}\frac{\partial^2 u}{\partial x^2} - \ell_{23}\frac{\partial^2 u}{\partial x\partial y}\right) -$$
$$-\frac{\partial^2}{\partial x\partial y}\left(\ell_{13}\frac{\partial^2 u}{\partial y^2} + \ell_{23}\frac{\partial^2 u}{\partial x^2} - \ell_{33}\frac{\partial^2 u}{\partial x\partial y}\right) +$$
$$+\frac{\partial^2}{\partial y^2}\left(\ell_{11}\frac{\partial^2 u}{\partial y^2} + \ell_{12}\frac{\partial^2 u}{\partial x^2} - \ell_{13}\frac{\partial^2 u}{\partial x\partial y}\right) = f .$$

The equilibrium problems for the plane system are translated into boundary value problems for the equation $Eu = f$. For instance, the Dirichlet problem corresponds to the physical problem of forces given on the boundary ∂A of A. Of course the solution of this problem must be of class C^2 in $\overline{A}$ since the continuity of the second derivatives of u corresponds to the continuity of $\widehat{xx}$, $\widehat{yy}$, $\widehat{xy}$.

In order to apply the theory developed in this paper we must only verify that E is positive elliptic. This is an immediate consequence of the positive-definiteness of the quadratic form $\ell_{ij}\lambda^i\lambda^j$.

NOTES

*The subscript z under the operator D means that this operates on $\mathscr{P}(z, \zeta)$ as a function of z.

§Prof. Agmon in his paper [1] concerning his double layer extension states: "A similar approach could also be used in the case of variable coefficients if a fundamental solution in the large is available". I wish to observe that when dealing with variable coefficients, more than a fundamental solution in the large is needed, in fact a principal fundamental solution must be used.

#The integral is extended over the arc of ∂A between z_0 and z described in the positive sense by a point starting from z_0 and going to z.

±This scalar product has obviously a different meaning from the scalar product introduced in section 2.

**We use the same notations as in Milne-Thomson [14].

BIBLIOGRAPHY

1. S. Agmon, Multiple layer potential and the Dirichlet problem for higher order elliptic equations in the plane, I - Comm. Pure Appl. Math., 1957.
2. F. E. Browder, The Dirichlet problem for linear elliptic equations of arbitrary even order with variable coefficients, Proc. Nat. Acad. Sci., U. S. A., 1952.
3. F. E. Browder, Assumption of boundary values and the Green's function in the Dirichlet problem for the general elliptic equation, Proc. Nat. Acad. Sci., U. S. A., 1953.
4. G. Fichera, Una introduzione alla teoria delle equazioni integrali singolari, Rend. di Matem. e delle sue Appl., 1958.
5. K. O. Friedrichs, On the differentiability of the solutions of linear elliptic differential equations, Comm. Pure Appl. Math., 1953.
6. L. Gårding, Dirichlet's problem for linear elliptic partial differential equations, Math. Scand., 1953.
7. G. Giraud, Généralisation des problèmes sur les opérations du type elliptique, Bull. Sci. Math., 1932.
8. L. Hörmander, Local and global properties of fundamental solutions, Math. Scand., 1957.
9. O. D. Kellogg, Foundations of potential theory, Springer-Verlag, 1929.
10. F. John, Plane waves and spherical means applied to partial differential equations, New York Interscience Publ., 1955.
11. P. D. Lax, On Cauchy's problem for hyperbolic equations and the differentiability of solutions of elliptic equations, Comm. Pure Appl. Math., 1955.
12. E. E. Levi, I problemi dei valori al contorno per le equazioni lineari totalmente ellittiche alle derivate parziali, Mem. Soc. Ital. dei XL, 1909.
13. Y. B. Lopatinskii, On a method of reducing boundary problems for a system of differential equations type to regular integral equations, Ukrain Mat. Z., 1953 (in Russian).
14. L. M. Milne-Thomson, Plane Elastic Systems, Ergeb. der Math., Springer-Verlag, 1960.
15. C. Miranda, Equazioni alle derivate parziali di tipo ellittico, Ergeb. der Math., Springer-Verlag, 1955.

16. C. Miranda, Teorema del massimo modulo e teorema di esistenza e di unicità per il problema di Dirichlet relativo alle equazioni ellittiche in due variabili, Ann. di Mat. pura e appl., 1958.

17. C. B. Morrey and L. Nirenberg, On the analicity of the solutions of partial differential equations, Comm. on pure and appl. Math., 1957.

18. N. I. Muskhelishvili, Singular integral equations, P. Noordhoff, Groningen, 1953.

19. M. Nicolesco, Les fonctions polyharmoniques, Actual. Sci. Ind. Hermann Paris, 1936.

20. L. Nirenberg, Remarks on strongly elliptic partial differential equations, Comm. pure Appl. Math., 1955.

21. M. Picone, Nuovi indirizzi di ricerca nella teoria e nel calcolo delle soluzioni di talune equazioni lineari alle derivate parziali della Fisica Matematica, Ann. Sc. Norm. Sup. Pisa, 1936.

22. A. Pleijel, On Green's functions for elastic plates with clamped, supported and free edges, Proc. Symp. on Spectral Theory and Diff. Problems, Oklahoma Agric. Mech. College, 1951.

23. D. I. Sherman, The static plane problem of the theory of elasticity for inhomogeneous media, Trud. seism. Inst. Ak. Nauk SSSR, 1938 (in Russian).

24. D. I. Sherman, The plane problem of the theory of elasticity for an anisotropic medium, Trud. seism. Inst. Ak. Nauk, 1938 (in Russian).

25. M. I. Visik, On strongly elliptic systems of differential equations, Mat. Sbornik, 1951 (in Russian).

ACKNOWLEDGMENT

The author wishes to express his acknowledgment to Prof. Milne-Thomson of the Mathematics Research Center, United States Army, for calling his attention to the boundary value problem of inhomogeneous anisotropic elasticity for plane systems, whose consideration originated the present paper. The necessity of obtaining for the physical problem solutions of the boundary value problems of elasticity with continuous derivatives on the boundary up to an order higher than that given by other methods, led the author to the procedures outlined in the present paper.