Chapter 11

<Solution>

1. Show that: $\epsilon^{rst} = g^{ri} g^{sj} g^{tk} \epsilon_{ijk}$

$$\begin{cases} \epsilon^{rst} = \dfrac{\partial \theta^r}{\partial x^d} \dfrac{\partial \theta^s}{\partial x^e} \dfrac{\partial \theta^t}{\partial x^f} e^{def} \\ \epsilon_{rst} = \dfrac{\partial x^d}{\partial \theta^r} \dfrac{\partial x^e}{\partial \theta^s} \dfrac{\partial x^f}{\partial \theta^t} e_{def} \end{cases}$$

$$g^{ri} g^{sj} g^{tk} \epsilon_{ijk} = \frac{\partial \theta^r}{\partial x^m} \frac{\partial \theta^i}{\partial x^m} \frac{\partial \theta^s}{\partial x^n} \frac{\partial \theta^j}{\partial x^n} \frac{\partial \theta^t}{\partial x^g} \frac{\partial \theta^k}{\partial x^g} \frac{\partial x^a}{\partial \theta^i} \frac{\partial x^b}{\partial \theta^j} \frac{\partial x^c}{\partial \theta^k} e_{abc}$$

$$= \frac{\partial \theta^r}{\partial x^m} \frac{\partial \theta^i}{\partial x^m} \frac{\partial x^a}{\partial \theta^i} \frac{\partial \theta^s}{\partial x^n} \frac{\partial \theta^j}{\partial x^n} \frac{\partial x^b}{\partial \theta^j} \frac{\partial \theta^t}{\partial x^g} \frac{\partial \theta^k}{\partial x^g} \frac{\partial x^c}{\partial \theta^k} e_{abc}$$

$$\doteq \frac{\partial \theta^r}{\partial x^m} \delta_{am} \frac{\partial \theta^s}{\partial x^n} \delta_{bn} \frac{\partial \theta^t}{\partial x^g} \delta_{cg} e_{abc}$$

$$= \frac{\partial \theta^r}{\partial x^a} \frac{\partial \theta^s}{\partial x^b} \frac{\partial \theta^t}{\partial x^c} e_{abc}$$

$$= \frac{\partial \theta^r}{\partial x^d} \frac{\partial \theta^s}{\partial x^e} \frac{\partial \theta^t}{\partial x^f} e^{def}$$

$$= \epsilon^{rst}$$

2. Show that: $\epsilon^{rst} \epsilon^{ijk} g_{ri} g_{sj} g_{tk} = 6$

$$\epsilon^{rst} \epsilon^{ijk} g_{ri} g_{sj} g_{tk}$$

$$= \frac{\partial \theta^r}{\partial x^a} \frac{\partial \theta^s}{\partial x^b} \frac{\partial \theta^t}{\partial x^c} e^{abc} \frac{\partial \theta^i}{\partial x^d} \frac{\partial \theta^j}{\partial x^e} \frac{\partial \theta^k}{\partial x^f} e^{def} \frac{\partial x^m}{\partial \theta^r} \frac{\partial x^m}{\partial \theta^i} \frac{\partial x^n}{\partial \theta^s} \frac{\partial x^n}{\partial \theta^j} \frac{\partial x^g}{\partial \theta^t} \frac{\partial x^g}{\partial \theta^k}$$

$$= \frac{\partial \theta^r}{\partial x^a} \frac{\partial x^m}{\partial \theta^r} \frac{\partial \theta^s}{\partial x^b} \frac{\partial x^n}{\partial \theta^s} \frac{\partial \theta^t}{\partial x^c} \frac{\partial x^g}{\partial \theta^t} \frac{\partial \theta^i}{\partial x^d} \frac{\partial x^m}{\partial \theta^i} \frac{\partial \theta^j}{\partial x^e} \frac{\partial x^n}{\partial \theta^j} \frac{\partial \theta^k}{\partial x^f} \frac{\partial x^g}{\partial \theta^k} e^{abc} e^{def}$$

$$= \frac{\partial x^m}{\partial x^a} \frac{\partial x^n}{\partial x^b} \frac{\partial x^g}{\partial x^c} \frac{\partial x^m}{\partial x^d} \frac{\partial x^n}{\partial x^e} \frac{\partial x^g}{\partial x^f} e^{abc} e^{def}$$

$$= \delta_{ma} \delta_{nb} \delta_{gc} \delta_{md} \delta_{ne} \delta_{gf} e^{abc} e^{def}$$

$$= \delta_{ad} \delta_{be} \delta_{cf} e^{abc} e^{def}$$

$$= e^{abc} \cdot e^{abc} = \delta_{bb} \delta_{cc} - \delta_{bc} \delta_{cb}$$

$$= 6$$

P. 593,
(11.25)

$$g_{ij} = \frac{\partial x_s}{\partial \theta^i}\frac{\partial x_s}{\partial \theta^j}$$

$$\frac{\partial g_{ij}}{\partial \theta^k} = \frac{\partial^2 x_s}{\partial \theta^i \partial \theta^k}\frac{\partial x_s}{\partial \theta^j} + \frac{\partial x_s}{\partial \theta^i}\frac{\partial^2 x_s}{\partial \theta^j \partial \theta^k} \qquad (a)$$

$$\frac{\partial g_{ik}}{\partial \theta^j} = \frac{\partial^2 x_s}{\partial \theta^i \partial \theta^j}\frac{\partial x_s}{\partial \theta^k} + \frac{\partial x_s}{\partial \theta^i}\frac{\partial^2 x_s}{\partial \theta^k \partial \theta^j} \qquad (b)$$

$$\frac{\partial g_{jk}}{\partial \theta^i} = \frac{\partial^2 x_s}{\partial \theta^j \partial \theta^i}\frac{\partial x_s}{\partial \theta^k} + \frac{\partial x_s}{\partial \theta^j}\frac{\partial^2 x_s}{\partial \theta^k \partial \theta^i} \qquad (c)$$

(b) + (c) − (a) :

$$2\frac{\partial^2 x_s}{\partial \theta^i \partial \theta^j}\frac{\partial x_s}{\partial \theta^k} - \cancel{\frac{\partial^2 x_s}{\partial \theta^i \partial \theta^k}\frac{\partial x_s}{\partial \theta^j}} + \cancel{\frac{\partial x_s}{\partial \theta^j}\frac{\partial^2 x_s}{\partial \theta^k \partial \theta^i}}$$

$$= 2\,\Gamma_{ijk}$$

Thus,

$$\Gamma_{ijk} = \frac{1}{2}\left[(b)+(c)-(a)\right] = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial \theta^j} + \frac{\partial g_{jk}}{\partial \theta^i} - \frac{\partial g_{ij}}{\partial \theta^k}\right]$$

(11.50)

p.548,

$$(11.45):\quad \tilde{u}_i = \sum_j g_{ij} \frac{u^{\langle j \rangle}}{\sqrt{g_{jj}}}$$

$$\tilde{u}_1 = \sum_j g_{1j} \frac{u^{\langle j \rangle}}{\sqrt{g_{jj}}}$$

$$\tilde{a}_1 = g_{11} \frac{a^{\langle 1 \rangle}}{\sqrt{g_{11}}} + g_{12} \frac{a^{\langle 2 \rangle}}{\sqrt{g_{22}}} + g_{13} \frac{a^{\langle 3 \rangle}}{\sqrt{g_{33}}}$$

$$= (1) \frac{a_r}{(1)} + (0) + 0 = a_r$$

$$\tilde{a}_2 = g_{21} \frac{a^{\langle 1 \rangle}}{\sqrt{g_{11}}} + g_{22} \frac{a^{\langle 2 \rangle}}{\sqrt{g_{22}}} + g_{23} \frac{a^{\langle 3 \rangle}}{\sqrt{g_{33}}}$$

$$= 0 + (r)^2 \frac{a_\theta}{r} + 0 = r a_\theta$$

$$\tilde{a}_3 = g_{31} \frac{a^{\langle 1 \rangle}}{\sqrt{g_{11}}} + g_{32} \frac{a^{\langle 2 \rangle}}{\sqrt{g_{22}}} + g_{33} \frac{a^{\langle 3 \rangle}}{\sqrt{g_{33}}}$$

$$= 0 + 0 + (1) \frac{a_z}{1} = a_z$$

p. 602.

$$ds^2 - dS^2 = 2\tilde{\gamma}_{ij}\, d\theta^i d\theta^j$$

A deformed material element is defined by $d\theta^i \underset{\sim}{g}_i$ (i not summed)

$$= d\theta^i \sqrt{g_{ii}} \frac{\underset{\sim}{g}_i}{\sqrt{g_{ii}}}$$

therefore, the physical component of the differential line element is $d\theta^i \sqrt{g_{ii}}$ which has the dimension of length

so, we may write

$$ds^2 - dS^2 = \sum_{i,j=1}^{3} 2 \frac{\tilde{\gamma}_{ij}}{\sqrt{g_{ii}}\sqrt{g_{jj}}} \left(\sqrt{g_{ii}}\, d\theta^i\right)\left(\sqrt{g_{jj}}\, d\theta^j\right)$$

where $\gamma_{\langle ij \rangle} = \dfrac{\tilde{\gamma}_{ij}}{\sqrt{g_{ii}}\sqrt{g_{jj}}}$ is dimensionless. (i, j not summed)

Cylindrical coordinates

$$\gamma_{\langle 11 \rangle} = \frac{\tilde{\gamma}_{11}}{\sqrt{g_{11}}\sqrt{g_{11}}} = \tilde{\gamma}_{11}$$

p. 604

The Lagrangian strain is given by

$2E_{RS} = u_{R;S} + u_{S;R} + u_{M;R}\, u^{M}{}_{;S}$ ← (11.93)

where u_R is the covariant component of the displacement vector and ";" denotes the covariant differentiation.

(a) Find the tensor components for the strain in the cylindrical coordinate system.

(b) Determine the corresponding physical components of strain tensor.

$$u_{R;S} = \frac{\partial u_R}{\partial x_S} - \Gamma^N_{RS} u_N \qquad u^M{}_{;S} = \frac{\partial u^M}{\partial x_S} + \Gamma^M_{RS} u^R$$

in the cylindrical coordinate system: (r, θ, z)

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} \qquad \Gamma^1_{22} = -r$$

the rest of Γ^i_{jk} are zero.

$$u_{1;1} = \frac{\partial u_1}{\partial r} - \Gamma^M_{11} u_M = \frac{\partial u_1}{\partial r}$$

$$u_{1;2} = \frac{\partial u_1}{\partial \theta} + \Gamma^M_{12} u_M = \frac{\partial u_1}{\partial \theta} - \frac{1}{r} u_2$$

$$u_{1;3} = \frac{\partial u_1}{\partial z}$$

$$u_{2;1} = \frac{\partial u_2}{\partial r} - \Gamma^2_{21} u_2 = \frac{\partial u_2}{\partial r} - \frac{1}{r} u_2$$

$$u_{2;2} = \frac{\partial u_2}{\partial \theta} - \Gamma^1_{22} u_1 = \frac{\partial u_2}{\partial \theta} + r u_1$$

$$u_{2;3} = \frac{\partial u_2}{\partial z}$$

$$u_{3;1} = \frac{\partial u_3}{\partial r}, \quad u_{3;2} = \frac{\partial u_3}{\partial \theta}, \quad u_{3;3} = \frac{\partial u_3}{\partial z}$$

$$u^r{}_{;s} = ?$$

$$u^1{}_{;1} = \frac{\partial u^1}{\partial r} + \cancelto{0}{\Gamma^1_{r1} u^r} = \frac{\partial u^1}{\partial r}$$

$$u^2{}_{;1} = \frac{\partial u^2}{\partial r} + \Gamma^2_{21} u^2 = \frac{\partial u^2}{\partial r} + \frac{1}{r} u^2$$

$$u^3{}_{;1} = \frac{\partial u^3}{\partial r}$$

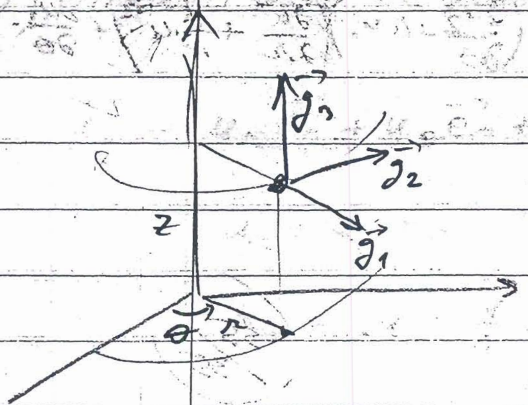
$$u^1{}_{;2} = \frac{\partial u^1}{\partial \theta} + \Gamma^1_{22} u^2 = \frac{\partial u^1}{\partial \theta} - r u^2$$

$$u^2{}_{;2} = \frac{\partial u^2}{\partial \theta} + \Gamma^2_{12} u^1 = \frac{\partial u^2}{\partial \theta} + \frac{1}{r} u^1$$

$$u^3{}_{;2} = \frac{\partial u^3}{\partial \theta}$$

$$u^1{}_{;3} = \frac{\partial u^1}{\partial z} \qquad u^2{}_{;3} = \frac{\partial u^2}{\partial z} \qquad u^3{}_{;3} = \frac{\partial u^3}{\partial z}$$

$$\vec{u} = u_1 \vec{g}^1 + u_2 \vec{g}^2 + u_3 \vec{g}^3 = u^1 \vec{g}_1 + u^2 \vec{g}_2 + u^3 \vec{g}_3$$



$$\vec{g}_1 = \vec{g}^1$$

$$\vec{g}_2 = r^2 \vec{g}^2$$

$$\vec{g}_3 = \vec{g}^3$$

$$2E_{11} = u_{1;1} + u_{1;1} + u_{m;1} u^m_{;1} = 2\frac{\partial u_1}{\partial r} + \frac{\partial u_1}{\partial r} \cdot \frac{\partial u^1}{\partial r} +$$

$$+ \left(\frac{\partial u_2}{\partial r} - \frac{1}{r} u_2\right)\left(\frac{\partial u^2}{\partial r} + \frac{1}{r} u^2\right) + \frac{\partial u_3}{\partial r} \cdot \frac{\partial u^3}{\partial r}$$

$$2E_{12} = u_{2;1} + u_{1;2} + u_{m;1} u^m_{;2} = \frac{\partial u_1}{\partial \theta} - \frac{1}{r} u_2 + \frac{\partial u_2}{\partial r} - \frac{1}{r} u_2$$

$$+ \frac{\partial u_1}{\partial r}\left(\frac{\partial u^1}{\partial \theta} - r u^2\right) + \left(\frac{\partial u_2}{\partial r} - \frac{1}{r} u_2\right)\left(\frac{\partial u^2}{\partial \theta} + \frac{1}{r} u^1\right) + \frac{\partial u_3}{\partial r} \cdot \frac{\partial u^3}{\partial \theta}$$

$$2E_{13} = u_{3;1} + u_{1;3} + u_{m;1} u^m_{;3} = \frac{\partial u_3}{\partial r} + \frac{\partial u_1}{\partial z} + \frac{\partial u_1}{\partial r} \frac{\partial u^1}{\partial z} + \left(\frac{\partial u_2}{\partial r} - \frac{1}{r} u_2\right) \frac{\partial u^2}{\partial z}$$

$$+ \frac{\partial u_3}{\partial r} \cdot \frac{\partial u^3}{\partial z}$$

$$E_{21} = u_{1;2} + u_{2;1} + u_{m;2}\, u^m_{;1} = \frac{\partial u_1}{\partial \theta} - \frac{1}{r}u_2 + \frac{\partial u_2}{\partial r} - \frac{1}{r}u_2$$

$$+ \left(\frac{\partial u_1}{\partial \theta} - \frac{1}{r}u_2\right)\frac{\partial u^1}{\partial r} + \left(\frac{\partial u_2}{\partial \theta} + r u_1\right)\left(\frac{\partial u^2}{\partial r} + \frac{1}{r}u^2\right) + \frac{\partial u_3}{\partial \theta}\cdot\frac{\partial u^3}{\partial r}$$

b)

$$E_{22} = 2u_{2;2} + u_{m;2}\, u^m_{;2} =$$

For small deformation, we neglect the nonlinear terms.

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$$E_{11} = \frac{\partial u_1}{\partial r}$$

$$E_{22} = \frac{\partial u_2}{\partial \theta} + r u_1$$

$$E_{33} = \frac{\partial u_3}{\partial z}$$

$$E_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial \theta} + \frac{\partial u_2}{\partial r}\right) - \frac{u_2}{r}$$

$$E_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial \theta}\right)$$

$$E_{31} = \frac{1}{2}\left(\frac{\partial u_3}{\partial r} + \frac{\partial u_1}{\partial z}\right)$$

P. 618

(11.169)

$$\overset{\circ}{\underset{\sim}{\tau}}{}^{(1)} = \overset{\circ}{\tau}{}^{ij} \underset{\sim}{g}_i \otimes \underset{\sim}{g}_j = \sigma^{*}_{ij} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$$

$$= \left(\overset{\circ}{\sigma}_{ij} - \frac{\partial v^i}{\partial x^k} \sigma_{kj} - \frac{\partial v^j}{\partial x^k} \sigma_{ik} \right) \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$$

$$= \frac{\partial x^i}{\partial \theta^r} \frac{\partial x^j}{\partial \theta^s} \overset{\circ}{\tau}{}^{rs} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$$

$$\overset{\circ}{\underset{\sim}{\tau}} = \overset{\circ}{\underset{\sim}{\tau}}{}^{(1)} + \left[\tau^{mj} \tilde{v}^i|_m + \tau^{im} \tilde{v}^j|_m \right] \underset{\sim}{g}_i \otimes \underset{\sim}{g}_j$$

$$= \overset{\circ}{\underset{\sim}{\tau}}{}^{(1)} + \tau^{ij} \overset{\circ}{\underset{\sim}{g}}_i \otimes \underset{\sim}{g}_j + \tau^{ij} \underset{\sim}{g}_i \otimes \overset{\circ}{\underset{\sim}{g}}_j$$

$$\overset{\circ}{\underset{\sim}{\tau}}{}^{(1)} = \left(\overset{\circ}{\sigma}_{ij} - \tau^{rs} \frac{\partial v^i}{\partial \theta^r} \frac{\partial x^j}{\partial \theta^s} - \tau^{rs} \frac{\partial x^i}{\partial \theta^r} \frac{\partial v^j}{\partial \theta^s} \right) \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$$

$$= \overset{\circ}{\sigma}_{ij} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j - \tau^{rs} \frac{\partial v^i}{\partial \theta^r} \underset{\sim}{e}_i \otimes \frac{\partial x^j}{\partial \theta^s} \underset{\sim}{e}_j - \tau^{rs} \frac{\partial x^i}{\partial \theta^r} \underset{\sim}{e}_i \otimes \frac{\partial v^j}{\partial \theta^s} \underset{\sim}{e}_j$$

$$= \overset{\circ}{\sigma}_{ij} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j - \tau^{rs} \overset{\circ}{\underset{\sim}{g}}_r \otimes \underset{\sim}{g}_s - \tau^{rs} \underset{\sim}{g}_r \otimes \overset{\circ}{\underset{\sim}{g}}_s$$

$\underset{\sim}{\tau} = \sigma_{ij} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$ ← fixed bases

$\overset{\circ}{\underset{\sim}{\tau}} = \overset{\circ}{\sigma}_{ij} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$

53:246 & 58:258 Continuum Mechanics and Plasticity
Exam #2, May 4, 2006

OPEN-BOOK, 75 Minutes
(Give all details of your derivations)

1. Consider the simple shear deformation in the $X_1 - X_2$ space. Use the hypoelastic constitutive equation with the Cotter-Rivlin stress rate to determine the expressions of the stress components, expressed in terms of the shear strain γ. (30 points)

2. In the case of uniaxial stress, the endochronic constitutive equation is

$$\sigma = E_0 \int_0^z \rho(z-z') \frac{d\varepsilon^p}{dz'} dz'$$

Consider the kernel function expressed by

$$E_0 \rho(z) = E_0 \delta(z) + E_1 e^{-\alpha z}$$

where E_0, E_1 and α are constants and δ is the Dirac delta function. Use also the isotropic-hardening function $f(z) = e^{\beta z}$.

(1) Derive the equation for stress σ expressed in terms of plastic strain ε^p.
(2) Explain how you would determine the material constants. Derive the equations for the determination of these constants.

(30 points)

3. The strain-displacement relations are given by

$$\tilde{\gamma}_{ij} = \frac{1}{2}\left(\tilde{U}_i\Big|_j^0 + \tilde{U}_j\Big|_i^0 + \tilde{U}^m\Big|_i^0 \tilde{U}_m\Big|_j^0 \right) \qquad \text{(a)}$$

Consider cylindrical coordinate system and answer the following questions:

(1) Find the expressions of the physical components $U^{\langle i \rangle}$.
(2) Express $\tilde{U}_i$ in terms of $U^{\langle i \rangle}$ and θ^i.
(3) Use Equation (a) to express component $\tilde{\gamma}_{11}$ in terms of $U^{\langle i \rangle}$ and θ^i.

(40 points)

(1) Cotter - Rivlin stress rate is

$$\sigma^* = \frac{D\sigma}{Dt} + \underset{\sim}{L}^T \cdot \underset{\sim}{\sigma} + \underset{\sim}{\sigma} \cdot \underset{\sim}{L} \qquad (7.106)$$

in $X_1 - X_2$ space simple shear may be written as,

$$x_1 = X_1 + 2\omega t X_2 \ , \ x_2 = X_2$$

$$u_1 = x_1 - X_1 = 2\omega t X_2 \ , \ u_2 = 0 \quad , \ \omega = const$$

$$[L] = \begin{bmatrix} 0 & 2\omega \\ 0 & 0 \end{bmatrix} \qquad [D] = \begin{bmatrix} 0 & \omega \\ \omega & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 2\omega + 0 \\ 0 + 2\omega & 0 \end{bmatrix}$$

the hypoelastic constitutive equation is

$$\sigma^*_{ij} = 2\mu D_{ij} \quad \text{and} \quad D_{kk} = 0$$

$$\begin{bmatrix} \dot\sigma_{11} & \dot\sigma_{12} \\ \dot\sigma_{12} & \dot\sigma_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2\omega & 0 \end{bmatrix}\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} + \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}\begin{bmatrix} 0 & 2\omega \\ 0 & 0 \end{bmatrix} = 2\mu\begin{bmatrix} 0 & \omega \\ \omega & 0 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} \dot\sigma_{11} & \dot\sigma_{12} \\ \dot\sigma_{12} & \dot\sigma_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2\omega\sigma_{11} & 2\omega\sigma_{12} \end{bmatrix} + \begin{bmatrix} 0 & 2\omega\sigma_{11} \\ 0 & 2\omega\sigma_{12} \end{bmatrix} = 2\mu\begin{bmatrix} 0 & \omega \\ \omega & 0 \end{bmatrix}$$

$\gamma = 2\omega t$

$\sigma_{12} = \mu\gamma$

$\sigma_{22} = -\mu\gamma^2$

$$\dot\sigma_{11} = 0 \checkmark, \quad \dot\sigma_{12} + 2\omega\sigma_{11} = 2\mu\omega \checkmark, \quad \dot\sigma_{22} = -4\omega\,\sigma_{12} \checkmark$$

$$\int \dot\sigma_{11} dt = \sigma_{11} \text{ const chosen for simplicity} = 0$$

$$\int \dot\sigma_{12} dt = \int 2\mu\omega\, dt = 2\mu\omega t \ , \quad \sigma_{12} = 2\mu\omega t$$

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$$\int \dot\sigma_{22}\, dt = -\int 8\omega^2\mu t = -4\omega^2\mu t^2 \ , \quad \sigma_{22} = -4\omega^2\mu t^2$$

(2)

$$\sigma = E_0 \int_0^z \rho(z-z') \frac{d\varepsilon^p}{dz'} dz', \qquad E_0 \rho(z) = E_0 \delta(z) + E_1 e^{-\alpha z}$$

$$d\zeta = |d\varepsilon^p|, \quad \frac{d\zeta}{dz} = f(z) = e^{\beta z} = 1+\beta\zeta, \quad \frac{d\varepsilon^p}{dz} = \frac{d\varepsilon^p}{d\zeta}\frac{d\zeta}{dz} = \frac{d\varepsilon^p}{d\zeta} f(z)$$

Loading:

$$\sigma = \int_0^z E_0 \delta(z-z') \frac{d\varepsilon^p}{dz'} dz' + E_1 \int_0^z e^{-\alpha(z-z')} \frac{d\varepsilon^p}{dz'} dz'$$

$$= E_0 \frac{d\varepsilon^p}{dz} + E_1 \int_0^z e^{-\alpha(z-z')} \underbrace{\frac{d\varepsilon^p}{d\zeta'}}_{=+1} \underbrace{\frac{d\zeta'}{dz'}}_{=e^{\beta z'}} dz'$$

$$= E_0 e^{\beta z} + E_1 e^{-\alpha z} \int_0^z e^{(\alpha+\beta) z'} dz'$$

$$= E_0 e^{\beta z} + \frac{E_1 e^{-\alpha z}}{\alpha+\beta} \left[e^{(\alpha+\beta) z'} \right]_0^z$$

$$= E_0 e^{\beta z} + \frac{E_1 e^{-\alpha z}}{\alpha+\beta} \left[e^{(\alpha+\beta) z} - 1 \right] \qquad n = 1 + \frac{\alpha}{\beta}, \quad e^{-(\alpha+\beta) z} = e^{-\beta n z}$$

$$= E_0 e^{\beta z} + \frac{E_1}{\alpha+\beta} \left[e^{\beta z} - e^{-\alpha z} \right] = E_0 e^{\beta z} + \frac{E_1 e^{\beta z}}{\alpha+\beta} \left[1 - e^{-(\alpha+\beta) z} \right]$$

$$= E_0 (1+\beta\zeta) + E_1 \frac{(1+\beta\zeta)}{\beta n} \left[1 - \frac{1}{(1+\beta\zeta)^n} \right]$$

$$\boxed{\begin{matrix} (1+\beta\varepsilon)^{-(n-1)} \\ \dfrac{-(n-1)(1+\beta\varepsilon)^{-n}}{\beta} \end{matrix}}$$

Loading from zero, $\zeta = \varepsilon^p$

$$\sigma = E_0 (1+\beta\varepsilon^p) + \frac{E_1}{\beta n} (1+\beta\varepsilon^p) \left[1 - \frac{1}{(1+\beta\varepsilon^p)^n} \right]$$

At $\varepsilon^p \to 0$, $\sigma_y = E_0$

At $\varepsilon^p \to \infty$, $\displaystyle \frac{d\sigma}{d\varepsilon} = E_0\beta + \frac{E_1}{n} + \left.\frac{E_1}{\beta n} \frac{(n-1)(1+\beta\varepsilon)^{-n}}{\beta}\right|_{\varepsilon\to\infty}$

$$= E_0\beta + \frac{E_1}{n} = E_t$$

Asymptote:

$$\sigma = E_0(1+\beta\varepsilon) + \frac{E_1}{\beta n}(1+\beta\varepsilon)$$

$$\sigma^0 = \sigma_y + \left.\frac{E_1}{\beta n}\right|_{\varepsilon\to 0}$$

$$\frac{E_1}{n} = (\sigma^0 - \sigma_y)\beta$$

$$E_t = \sigma_y\beta + (\sigma^0 - \sigma_y)\beta = \sigma^0\beta \quad , \quad \beta = \frac{E_t}{\sigma^0}$$

(3)

$$\tilde{\gamma}_{ij} = \frac{1}{2}\left(\tilde{U}_i\big|_j^0 + \tilde{U}_j\big|_i^0 + \tilde{U}^m\big|_i^0 \tilde{U}_m\big|_j^0\right)$$

Cylindrical

$$\underset{\sim}{u} = \tilde{U}^i \underset{\sim}{G}_i = \tilde{U}_i \underset{\sim}{G}^i, \qquad G_{ij} = \begin{bmatrix}1&0&0\\0&(\theta^1)^2&0\\0&0&1\end{bmatrix}, \quad G^{ij} = \begin{bmatrix}1&0&0\\0&(\theta^1)^{-2}&0\\0&0&1\end{bmatrix}$$

$$\tilde{U}^{\langle i\rangle} = \tilde{U}^i \sqrt{G_{\underline{ii}}}, \qquad U^{\langle 1\rangle} = \tilde{U}^1\sqrt{G_{11}} = \tilde{U}^1, \quad U^{\langle 2\rangle} = \tilde{U}^2 \sqrt{G_{22}} = \tilde{U}^2\theta^1,$$

$$U^{\langle 3\rangle} = \tilde{U}^3\sqrt{G_{33}} = \tilde{U}^3$$

$$\tilde{U}^j\big|_i^0 = \frac{\partial \tilde{U}^j}{\partial \theta^i} + {}_0\Gamma^j_{ki}\tilde{U}^k \qquad \tilde{U}_i = \sum_j G_{ij}\frac{U^{\langle j\rangle}}{\sqrt{G_{jj}}}$$

$$\tilde{U}_j\big|_i^0 = \frac{\partial \tilde{U}_j}{\partial \theta^i} - {}_0\Gamma^k_{ji}\tilde{U}_k$$

$$\tilde{\gamma}_{11} = \frac{1}{2}\left(\tilde{U}_1\big|_1^0 + \tilde{U}_1\big|_1^0 + \tilde{U}^1\big|_1^0\tilde{U}_1\big|_1^0 + \tilde{U}^2\big|_1^0\tilde{U}_2\big|_1^0 + \tilde{U}^3\big|_1^0\tilde{U}_3\big|_1^0\right)$$

$$\tilde{U}_1\big|_1^0 = \frac{\partial \tilde{U}_1}{\partial \theta^1} - \cancelto{0}{{}_0\Gamma^1_{11}}\tilde{U}_1 - \cancelto{0}{{}_0\Gamma^2_{11}}\tilde{U}_2 - \cancelto{0}{{}_0\Gamma^3_{11}}\tilde{U}_3 = \frac{\partial \tilde{U}_1}{\partial \theta^1}$$

$$\tilde{U}^1\big|_1^0 = \frac{\partial \tilde{U}^1}{\partial \theta^1} + \cancelto{0}{{}_0\Gamma^1_{11}}\tilde{U}^1 + \cancelto{0}{{}_0\Gamma^1_{21}}\tilde{U}^2 + \cancelto{0}{{}_0\Gamma^1_{31}}\tilde{U}^3 = \frac{\partial \tilde{U}^1}{\partial \theta^1}$$

$$\tilde{U}^2\big|_1^0 = \frac{\partial \tilde{U}^2}{\partial \theta^1} + \cancelto{0}{{}_0\Gamma^2_{11}}\tilde{U}^1 + {}_0\Gamma^2_{21}\tilde{U}^2 + \cancelto{0}{{}_0\Gamma^2_{31}}\tilde{U}^3 = \frac{\partial \tilde{U}^2}{\partial \theta^1} + \frac{\tilde{U}^2}{\theta^1}$$

$$\tilde{U}_2\big|_1^0 = \frac{\partial \tilde{U}_2}{\partial \theta^1} - \cancelto{0}{{}_0\Gamma^1_{21}}\tilde{U}_1 - {}_0\Gamma^2_{21}\tilde{U}_2 - \cancelto{0}{{}_0\Gamma^3_{21}}\tilde{U}_3 = \frac{\partial \tilde{U}_2}{\partial \theta^1} - \frac{\tilde{U}_2}{\theta^1}$$

$$\tilde{U}^3\big|_1^0 = \frac{\partial \tilde{U}^3}{\partial \theta^1} + \cancelto{0}{{}_0\Gamma^3_{11}}\tilde{U}^1 + \cancelto{0}{{}_0\Gamma^3_{21}}\tilde{U}^2 + \cancelto{0}{{}_0\Gamma^3_{31}}\tilde{U}^3 = \frac{\partial \tilde{U}^3}{\partial \theta^1}$$

$$\tilde{U}_3\big|_1^0 = \frac{\partial \tilde{U}_3}{\partial \theta^1} - \cancelto{0}{{}_0\Gamma^1_{31}}\tilde{U}_1 - \cancelto{0}{{}_0\Gamma^2_{31}}\tilde{U}_2 - \cancelto{0}{{}_0\Gamma^3_{31}}\tilde{U}_3 = \frac{\partial \tilde{U}_3}{\partial \theta^1}$$

$$\tilde{\gamma}_{11} = \frac{1}{2}\left\{2\frac{\partial \tilde{U}_1}{\partial \theta^1} + \frac{\partial \tilde{U}^1}{\partial \theta^1}\frac{\partial \tilde{U}_1}{\partial \theta^1} + \left(\frac{\partial \tilde{U}^2}{\partial \theta^1} + \frac{\tilde{U}^2}{\theta^1}\right)\left(\frac{\partial \tilde{U}_2}{\partial \theta^1} - \frac{\tilde{U}_2}{\theta^1}\right) + \frac{\partial \tilde{U}^3}{\partial \theta^1}\frac{\partial \tilde{U}_3}{\partial \theta^1}\right\}$$

$$\tilde{U}_i = \sum_j G_{ij} \frac{U^{\langle j \rangle}}{\sqrt{G_{ij}}}$$

$$\tilde{U}_1 = G_{11} \frac{U^{\langle 1 \rangle}}{\sqrt{G_{11}}} + \cancel{G_{12}}^{\,0} \frac{U^{\langle 2 \rangle}}{\sqrt{G_{22}}} + \cancel{G_{13}}^{\,0} \frac{U^{\langle 3 \rangle}}{\sqrt{G_{33}}} = U^{\langle 1 \rangle}$$

$$\tilde{U}_2 = \cancel{G_{21}}^{\,0} \frac{U^{\langle 1 \rangle}}{\sqrt{G_{11}}} + G_{22} \frac{U^{\langle 2 \rangle}}{\sqrt{G_{22}}} + \cancel{G_{23}}^{\,0} \frac{U^{\langle 3 \rangle}}{\sqrt{G_{33}}} = \theta^1 U^{\langle 2 \rangle}$$

$$\tilde{U}_3 = \cancel{G_{31}}^{\,0} \frac{U^{\langle 1 \rangle}}{\sqrt{G_{11}}} + \cancel{G_{32}}^{\,0} \frac{U^{\langle 2 \rangle}}{\sqrt{G_{22}}} + G_{33} \frac{U^{\langle 3 \rangle}}{\sqrt{G_{33}}} = U^{\langle 3 \rangle}$$

$$\tilde{\gamma}_{11} = \frac{1}{2} \left\{ 2 \frac{\partial U^{\langle 1 \rangle}}{\partial \theta^1} + \frac{\partial U^{\langle 1 \rangle}}{\partial \theta^1} \frac{\partial U^{\langle 1 \rangle}}{\partial \theta^1} + \left[\frac{\partial \left(\frac{U^{\langle 2 \rangle}}{\theta^1} \right)}{\partial \theta^1} + \frac{U^{\langle 2 \rangle}}{(\theta^1)^2} \right] \left[\frac{\partial (\theta^1 U^{\langle 2 \rangle})}{\partial \theta^1} - U^{\langle 2 \rangle} \right] \right.$$

$$\left. + \frac{\partial U^{\langle 3 \rangle}}{\partial \theta^1} \frac{\partial U^{\langle 3 \rangle}}{\partial \theta^1} \right\}$$

$$\frac{\partial}{\partial \theta^1} \left(\frac{U^{\langle 2 \rangle}}{\theta^1} \right) = \frac{\theta^1 \frac{\partial U^{\langle 2 \rangle}}{\partial \theta^1} - U^{\langle 2 \rangle}}{(\theta^1)^2} = \frac{1}{\theta^1} \frac{\partial U^{\langle 2 \rangle}}{\partial \theta^1} - \frac{U^{\langle 2 \rangle}}{(\theta^1)^2}$$

$$\frac{\partial}{\partial \theta^1} \left(\theta^1 U^{\langle 2 \rangle} \right) = \theta^1 \frac{\partial U^{\langle 2 \rangle}}{\partial \theta^1} + U^{\langle 2 \rangle}$$

$$\tilde{\gamma}_{11} = \frac{1}{2} \left\{ 2 \frac{\partial U^{\langle 1 \rangle}}{\partial \theta^1} + \left(\frac{\partial U^{\langle 1 \rangle}}{\partial \theta^1} \right)^2 + \left(\frac{1}{\theta^1} \frac{\partial U^{\langle 2 \rangle}}{\partial \theta^1} \right) \left(\theta^1 \frac{\partial U^{\langle 2 \rangle}}{\partial \theta^1} \right) \right\}$$

$$= \frac{\partial U^{\langle 1 \rangle}}{\partial \theta^1} + \frac{1}{2} \left\{ \left(\frac{\partial U^{\langle 1 \rangle}}{\partial \theta^1} \right)^2 + \left(\frac{\partial U^{\langle 2 \rangle}}{\partial \theta^1} \right)^2 \right\}$$

53:246 (58:258) Continuum Mechanics and Plasticity

MID-TERM EXAM (4/30/02)

1.5 Hours, Open Class Notes

Name ___________________

1. The position vector of a point $\vec{r}$ is related to the spherical coordinates (R, α,θ) by
$$\vec{r} = \mathbf{R}(\sin\alpha\cos\theta)\,\vec{e}_1 + \mathbf{R}(\sin\alpha\sin\theta)\,\vec{e}_2 + \mathbf{R}(\cos\alpha)\,\vec{e}_3$$
Find the expressions of the covariant and contravariant base vectors of the spherical coordinate system. Find also the covariant metric tensor g_{ij}. (30 pts)

2. We recommend the use of contravariant true stress τ^{ij} and covariant strain γ_{ij}. Discuss the rationale for this choice of stress and strain. (20 pts)

3. This problem refers to the simple shear problem discussed in the class, which has non-zero normal stress components. In the case of plane stress, the von Mises yield criterion is given by

$$\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2 = Y^2$$

in the Cartesian coordinate system.

(a) Rewrite this yield criterion by use of the physical components of the contravariant true stress τ^{ij}. (5 pts)

(b) Find the expression of this criterion in terms of the tensorial components of τ^{ij}. (15 pts)

(c) Find the expression of this criterion in terms of the Cauchy stress components σ_{ij}. (15 pts)

(d) Sketch the yield surfaces in the σ_{12} *vs.* σ_{22} space with $\sigma_{\cancel{12}} = 0$, for values of shear K = 0, 0.4 and 1.0. (15 pts)
11

Spherical coordinates (R, α, θ)

$$\vec{g}_1 = \sin\alpha \cos\theta\, \vec{e}_1 + \sin\alpha \sin\theta\, \vec{e}_2 + \cos\alpha\, \vec{e}_3$$

$$\vec{g}_2 = R(\cos\alpha \cos\theta)\, \vec{e}_1 + R(\cos\alpha \sin\theta)\, \vec{e}_2 - R\sin\alpha\, \vec{e}_3$$

$$\vec{g}_3 = -R(\sin\alpha \sin\theta)\, \vec{e}_1 + R(\sin\alpha \cos\theta)\, \vec{e}_2$$

$g_{11} = \vec{g}_1 \cdot \vec{g}_1$, etc.

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & R^2 \sin^2\alpha \end{bmatrix}$$

Orthogonal coord. system

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^{-2} & 0 \\ 0 & 0 & R^{-2} \sin^{-2}\alpha \end{bmatrix}$$

$$\vec{g}^i = g^{ij}\, \vec{g}_j$$

$$\vec{g}^1 = g^{11}\, \vec{g}_1 = \vec{g}_1$$

$$\vec{g}^2 = g^{22}\, \vec{g}_2 = R^{-2}\, \vec{g}_2$$

$$\vec{g}^3 = g^{33}\, \vec{g}_3 = R^{-2} \sin^{-2}\alpha\, \vec{g}_3$$

2. Contravariant true stress is used because:

- τ^{ij} is easily related to the stress vector ✓
- if τ_{ij} is used the deformed area used is a different portion of the material than the undeformed, this is especially important for anisotropic materials ✓
- τ^{ij} is a "true stress" refferring to current/deformed configuration

Covariant Strain is used because:

- it has clear physical meaning, diagonal terms are related to relative elongation along coordinate curves, off diagonal related to angle changes ✓
- it uses imbedded coordinates & follows same material ✓

- The two are used in conjunction because when multiplied they result in work, and need one covariant and one contravariant to multiply. ✓

20/20

② Mathematically, contravariant and covariant tensor components, are equally justified. However from a physical stand point, a contravariant stress, and a covariant strain have the simplest physical interpretation and the simplest formulation respectively. Page 65
2:R of the text mentions that "the covariant stress cannot be related to the stress vectors in a simple way." In addition, the covariant stress does not have the same material element before and after deformation. This isn't acceptable usually but may be okay if the material isotropic. ~~Contravariant and covariant stress components match the material element but mixed stress components do not.~~ This is also a physical abnormality. Therefore for stress components, contravariant are the simplest and
2:22 provide the most straightforward interpretation. The covariant strain γ_{ij} has physical meaning: γ_{11}, γ_{22}, γ_{33} are the extensions of a line element along coordinate curves. The off-diagonal terms are related to the angle of shear. In addition the

differential strain energy, $dW = \sqrt{\frac{G}{g}}\,\tau^{ij} d\gamma_{ij}$. Which can be related as $\sqrt{\frac{G}{g}}\,\tau^{ij} = \frac{\partial W}{\partial \gamma_{ij}}$.

20/20

(3) shear problem from chapter 3.

$$G_{ik} = \begin{bmatrix} 1 & k & 0 \\ k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G^{ik} = \begin{bmatrix} 1+k^2 & -k & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Prob. (i)

$$\underset{\sim}{g}_K = \frac{\partial x_i}{\partial \theta^K} \underset{\sim}{e}_i$$

$$\underset{\sim}{g}_i \cdot \underset{\sim}{g}^j = \delta_i^j$$

substituting,

$$\frac{\partial x_j}{\partial \theta^i} \underset{\sim}{e}_j \cdot \underset{\sim}{g}^K = \delta_i^K$$

$$\frac{\partial \theta^i}{\partial x_m} \frac{\partial x_j}{\partial \theta^i} \underset{\sim}{e}_j \cdot \underset{\sim}{g}^K = \delta_i^K \frac{\partial \theta^i}{\partial x_m} = \frac{\partial \theta^K}{\partial x_m}$$

$$\delta_m^j \underset{\sim}{e}_j \cdot \underset{\sim}{g}^K = \frac{\partial \theta^K}{\partial x_m}$$

$\underset{\sim}{e}_m \cdot \underset{\sim}{g}^K = \dfrac{\partial \theta^K}{\partial x_m}$ = components of $\underset{\sim}{g}^K$ projected on $\underset{\sim}{e}_m$.

$$\therefore \underset{\sim}{g}^K = \frac{\partial \theta^K}{\partial x_m} \underset{\sim}{e}_m$$

Problem (2)

Deformation gradient

$$\underset{\sim}{F} = \underset{\sim}{g}_j \otimes G^j$$

From the transformation from Cartesian base vectors to covariant and contravariant base vectors, we have.

$$\underset{\sim}{F} = \frac{\partial x_i}{\partial \theta^j} \underset{\sim}{e}_i \otimes \frac{\partial \theta^j}{\partial X_K} \underset{\sim}{E}_K$$

$$= \frac{\partial x_i}{\partial X_K} \underset{\sim}{e}_i \otimes E_K$$

$$= F_{iK} \underset{\sim}{e}_i \otimes \underset{\sim}{E}_K \quad \checkmark$$

Problem (3)

Show that $\underset{\sim}{F}^{-1} = \underset{\sim}{G}_i \otimes \underset{\sim}{g}^i$, $\underset{\sim}{F}^{-T} = \underset{\sim}{g}^i \otimes \underset{\sim}{G}_i$

Proof: Deformation gradient $\underset{\sim}{F}$ transform $\underset{\sim}{G}_i$ into $\underset{\sim}{g}_i$, where $\underset{\sim}{G}_i$ and $\underset{\sim}{g}_i$ are covariant base vector in undeformed and deformed configuration, respectively.

$$\underset{\sim}{g}_i = \underset{\sim}{F} \cdot \underset{\sim}{G}_i \qquad (1)$$

we know

$$\underset{\sim}{F} = \underset{\sim}{g}_j \otimes \underset{\sim}{G}^j \qquad (2)$$

(1) $\Rightarrow$

$$\underset{\sim}{F}^{-1} \cdot \underset{\sim}{g}_i = \underset{\sim}{G}_i$$

proof $\rightarrow$ $\frac{\partial X_K}{\partial \theta^i} \underset{\sim}{e}_K \otimes \frac{\partial \theta^i}{\partial x_m} \underset{\sim}{e}_m$

$$\Rightarrow \underset{\sim}{F}^{-1} \cdot (\underbrace{\underset{\sim}{g}_i \otimes \underset{\sim}{g}^i}_{=\underset{\sim}{I}}) = \underset{\sim}{G}_i \otimes \underset{\sim}{g}^i$$

$$= \frac{\partial X_K}{\partial \theta^i} \frac{\partial \theta^i}{\partial x_m} \underset{\sim}{e}_K \otimes \underset{\sim}{e}_m$$

$$= \delta_{Km} \underset{\sim}{e}_K \otimes \underset{\sim}{e}_m$$

$$\Rightarrow \underset{\sim}{F}^{-1} = \underset{\sim}{G}_i \otimes \underset{\sim}{g}^i$$

$$= \underset{\sim}{e}_K \otimes \underset{\sim}{e}_K = \underset{\sim}{I}$$

In addition, $\underset{\sim}{F}^{-T} = (\underset{\sim}{F}^{-1})^T = (\underset{\sim}{F}^T)^{-1} = \underset{\sim}{g}^i \otimes \underset{\sim}{G}_i \checkmark$

Prob.
(4) (11.48)

$$\frac{\partial \bar{\underset{\sim}{g}}_i}{\partial \bar{\theta}^j} = \bar{\Gamma}^m_{ij}\, \bar{\underset{\sim}{g}}_m \quad \text{---(1)} \quad , \quad \bar{\underset{\sim}{g}}_i = \frac{\partial \theta^m}{\partial \bar{\theta}^i}\, \underset{\sim}{g}_m$$

$$\frac{\partial \bar{\underset{\sim}{g}}_i}{\partial \bar{\theta}^j} = \frac{\partial}{\partial \bar{\theta}^j}\left(\frac{\partial \theta^m}{\partial \bar{\theta}^i}\, \underset{\sim}{g}_m \right)$$

$$= \frac{\partial^2 \theta^m}{\partial \bar{\theta}^i \partial \bar{\theta}^j} \cdot \underset{\sim}{g}_m + \frac{\partial \theta^m}{\partial \bar{\theta}^i} \frac{\partial \underset{\sim}{g}_m}{\partial \bar{\theta}^j} \quad \text{----(2)}$$

But $\dfrac{\partial \underset{\sim}{g}_m}{\partial \bar{\theta}^j} = \dfrac{\partial \underset{\sim}{g}_m}{\partial \theta^s} \dfrac{\partial \theta^s}{\partial \bar{\theta}^j} = \Gamma^r_{ms}\, \underset{\sim}{g}_r \dfrac{\partial \theta^s}{\partial \bar{\theta}^j}$

Thus,

(2): $$\frac{\partial \bar{\underset{\sim}{g}}_i}{\partial \bar{\theta}^j} = \frac{\partial^2 \theta^m}{\partial \bar{\theta}^i \partial \bar{\theta}^j}\, \underset{\sim}{g}_m + \frac{\partial \theta^m}{\partial \bar{\theta}^i} \cdot \Gamma^r_{ms} \frac{\partial \theta^s}{\partial \bar{\theta}^j}\, \underset{\sim}{g}_r$$

$$= \frac{\partial^2 \theta^s}{\partial \bar{\theta}^i \partial \bar{\theta}^j}\, \underset{\sim}{g}_s + \Gamma^r_{ms} \frac{\partial \theta^m}{\partial \bar{\theta}^i} \frac{\partial \theta^s}{\partial \bar{\theta}^j} \frac{\partial \bar{\theta}^n}{\partial \theta^r}\, \bar{\underset{\sim}{g}}_n$$

$$= \frac{\partial^2 \theta^s}{\partial \bar{\theta}^i \partial \bar{\theta}^j} \frac{\partial \bar{\theta}^m}{\partial \theta^s}\, \bar{\underset{\sim}{g}}_m + \Gamma^t_{rs} \frac{\partial \theta^r}{\partial \bar{\theta}^i} \frac{\partial \theta^s}{\partial \bar{\theta}^j} \frac{\partial \bar{\theta}^m}{\partial \theta^t}\, \bar{\underset{\sim}{g}}_m \quad \text{---(3)}$$

(1) = (3):

$$\bar{\Gamma}^m_{ij} = \Gamma^t_{rs} \frac{\partial \theta^r}{\partial \bar{\theta}^i} \frac{\partial \theta^s}{\partial \bar{\theta}^j} \frac{\partial \bar{\theta}^m}{\partial \theta^t} + \frac{\partial^2 \theta^s}{\partial \bar{\theta}^i \partial \bar{\theta}^j} \frac{\partial \bar{\theta}^m}{\partial \theta^s}$$

Problem
(5) Show

$$\underset{\sim}{H} = \tilde{H}^{ij} g_i \otimes g_j \text{ is } \tilde{H}^{ij}\big|_k = \frac{\partial \tilde{H}^{ij}}{\partial \theta^k} + \tilde{H}^{mj} \Gamma^i_{mk} + \tilde{H}^{im} \Gamma^j_{mk}$$

$$\frac{\partial \underset{\sim}{H}}{\partial \theta^r} = \frac{\partial(\tilde{H}^{ij} g_i \otimes g_j)}{\partial \theta^r} = \frac{\partial \tilde{H}^{ij}}{\partial \theta^r} g_i \otimes g_j + \tilde{H}^{ij} \frac{\partial g_i}{\partial \theta_r} \otimes g_j + \tilde{H}^{ij} g_i \otimes \frac{\partial g_j}{\partial \theta_r}$$

$$= \underbrace{\left(\frac{\partial \tilde{H}^{ij}}{\partial \theta^r} g_i \otimes g_j + \Gamma^i_{rm} \tilde{H}^{mj} + \tilde{H}^{im} \Gamma^j_{rm} \right)}_{\tilde{H}^{ij}|_k} g_i \otimes g_j$$

✓

Problem
(6) Show

$$(v^k + w^k)\big|_i = v^k\big|_i + w^k\big|_i$$

$$(v^k + w^k)\big|_i = \frac{\partial}{\partial \theta^i}(v^k + w^k) + (v^j + w^j)\, \Gamma^k_{ji}$$

$$= \frac{\partial}{\partial \theta^i}(v^k) + v^j \Gamma^k_{ji} + \frac{\partial}{\partial \theta^i}(w^k) + w^j \Gamma^k_{ji}$$

$$= v^k\big|_i + w^k\big|_i$$

✓

Problem (7)

$$(v^j w^k)\Big|_i = \frac{\partial}{\partial\theta^i}(v^j w^k) + \Gamma^j_{im} v^m w^k + \Gamma^k_{im} v^j w^m$$

$$= \left(\frac{\partial v^j}{\partial\theta^i} w^k + \Gamma^j_{im} v^m w^k\right) + \left(\frac{\partial w^k}{\partial\theta^i} v^j + \Gamma^k_{im} v^j w^m\right)$$

$$= v^j\big|_i w^k + v^j w^k\big|_i$$

Problem (8)

$$\underset{\sim}{S} = \tilde{S}^{ij}\, \underset{\sim}{g}_i \otimes \underset{\sim}{g}_j = J\tau^{ij}\, \underset{\sim}{g}_i \otimes \underset{\sim}{g}_j \quad , \quad J = 1 \text{ for Simple Shear}$$

$$\therefore\ \tilde{S}^{ij} = \tau^{ij}$$

$$\frac{1}{2}\tilde{S}'^{\langle ij\rangle}\tilde{S}'^{\langle ij\rangle} = \frac{1}{3}Y^2$$

$$\tilde{S}'^{\langle ij\rangle} = J\tau'^{\langle ij\rangle} = J\tau^{\langle ij\rangle} - Jp\,g^{ij}\frac{\sqrt{g_{jj}}}{\sqrt{g^{ii}}} \quad (i, j \text{ not summed})$$

$$\left.\begin{aligned}
\tilde{S}'^{\langle 11\rangle} &= J\tau'^{\langle 11\rangle} = \frac{1}{\sqrt{1+K^2}}\left[\tilde{S}^{11} - Jp(1+K^2)\right] \\
\tilde{S}'^{\langle 22\rangle} &= J\tau'^{\langle 22\rangle} = \sqrt{1+K^2}\,(\tilde{S}^{22} - Jp) \\
\tilde{S}'^{\langle 33\rangle} &= -Jp\,,\quad \tilde{S}'^{\langle 12\rangle} = \tilde{S}^{12} + JpK \\
\tilde{S}'^{\langle 13\rangle} &= \tilde{S}'^{\langle 23\rangle} = \tilde{S}^{33} = \tilde{S}^{13} = \tilde{S}^{23} = 0
\end{aligned}\right\} \quad (1)$$

Mises yield criterion

$$\frac{1}{2}\left\{\left(\tilde{S}'^{\langle 11\rangle}\right)^2 + \left(\tilde{S}'^{\langle 22\rangle}\right)^2 + \left(\tilde{S}'^{\langle 33\rangle}\right)^2 + 2\left(\tilde{S}'^{\langle 12\rangle}\right)^2\right\} = \frac{1}{3}Y^2 \quad \cdots (2)$$

Substituting (1) into (2) and making use of (11.196), we obtain

$$C_{11}\sigma_{11}^2 + C_{1122}\sigma_{11}\sigma_{22} + C_{22}\sigma_{22}^2 + C_{1211}\sigma_{12}\sigma_{11} + C_{1222}\sigma_{12}\sigma_{22}$$
$$+ C_{12}\sigma_{12}^2 = \frac{1}{3}Y^2 \qquad (3)$$

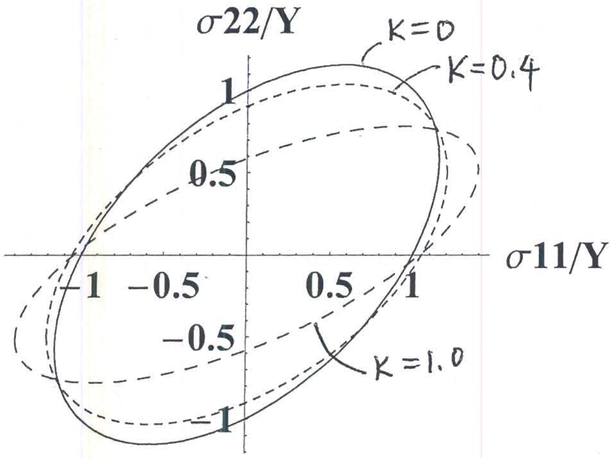
where the coefficients are functions of K. In the case of $\sigma_{12}=0$, the yield curve is

$$\left(\frac{1}{9} + \frac{K^2}{6} + \frac{2}{9(1+K^2)} - \frac{2K^2}{9(1+K^2)} + \frac{K^4}{18(1+K^2)}\right)\sigma_{11}^2$$
$$+\left(-\frac{1}{9} - \frac{2K^2}{3} - \frac{2}{9(1+K^2)} + \frac{5K^2}{9(1+K^2)} - \frac{2K^4}{9(1+K^2)}\right)\sigma_{11}\sigma_{22}$$
$$+\left(\frac{5}{18} + \frac{2K^2}{3} + \frac{1}{18(1+K^2)} - \frac{2K^2}{9(1+K^2)} + \frac{2K^4}{9(1+K^2)}\right)\sigma_{22}^2 = \frac{1}{3}Y^2 \cdots (4)$$

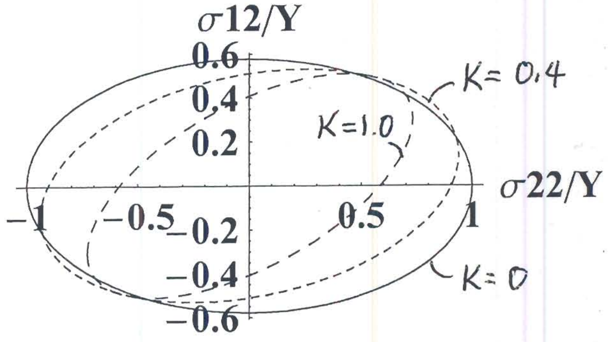
In the case of $\sigma_{11}=0$, the yield curve is

$$\left(1+\frac{2K^2}{1+K^2}\right)\sigma_{12}^2 + \left(\frac{5}{18} + \frac{2K^2}{3} + \frac{1}{18(1+K^2)} - \frac{2K^2}{9(1+K^2)} + \frac{2K^4}{9(1+K^2)}\right)\sigma_{22}^2$$
$$+\left(-\frac{4K}{3} + \frac{2K}{3(1+K^2)} - \frac{4K^3}{3(1+K^2)}\right)\sigma_{12}\sigma_{22} = \frac{1}{3}Y^2 \cdots (5)$$

(4) and (5) may be plotted for different values of K's. The curves are similar to those of Fig. 11.12. (4) and (5) are not quite the same as (11.197) and (11.198) because equ.(2) is different from (11.193). (3) may be compared to (11.192) which is the Mises yield criterion defined by the Cauchy Stress.



$\sigma_{12} = 0$



$\sigma_{11} = 0$

Problem (9)

From (11.138) the 2nd P-K stress is

$$\underset{\sim}{\Pi} = \tilde{S}^{ij}\, \underset{\sim}{G}_i \otimes \underset{\sim}{G}_j = J\tau^{ij}\, \underset{\sim}{G}_i \otimes \underset{\sim}{G}_j \quad , \quad J = 1 \text{ for simple shear.}$$

$$= \tilde{\Pi}^{ij}\, \underset{\sim}{G}_i \otimes \underset{\sim}{G}_j$$

$$\underset{\sim}{\Pi} = \tilde{\Pi}^{ij} \sqrt{G_{ii}}\sqrt{G_{jj}}\, \frac{\underset{\sim}{G}_i}{\sqrt{G_{ii}}} \otimes \frac{\underset{\sim}{G}_j}{\sqrt{G_{jj}}} = \tilde{\Pi}^{\langle ij\rangle}\, \underset{\sim}{E}_i \otimes \underset{\sim}{E}_j$$

$$\tilde{\Pi}^{\langle ij\rangle} = \tilde{\Pi}^{ij}\sqrt{G_{ii}}\sqrt{G_{jj}}$$

Orthogonal coordinate system

$$G_{ij} = G^{ij} = \delta_{ij} \quad , \quad G = 1$$

$$\tilde{\Pi}^{\langle ij\rangle} = \tilde{\Pi}^{ij} = \Pi^{ij} \quad \text{with } \underset{\sim}{E}_i = \underset{\sim}{e}_i .$$

Mises yield criterion

$$\frac{1}{2}\left\{\left(\tilde{\pi}'^{\langle 11\rangle}\right)^2+\left(\tilde{\pi}'^{\langle 22\rangle}\right)^2+\left(\tilde{\pi}'^{\langle 33\rangle}\right)^2+2\left(\tilde{\pi}'^{\langle 12\rangle}\right)^2\right\}=\frac{1}{3}Y^2$$

$$\frac{1}{2}\left\{\left(\pi'^{11}\right)^2+\left(\pi'^{22}\right)^2+\left(\pi'^{33}\right)^2+2\left(\pi'^{12}\right)^2\right\}=\frac{1}{3}Y^2$$

$$\underset{\sim}{F}=\begin{bmatrix}1 & K & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \quad \underset{\sim}{F}^{-1}=\begin{bmatrix}1 & -K & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$[\pi]=J\,[F]^{-1}[\sigma][F]^{-T}$$

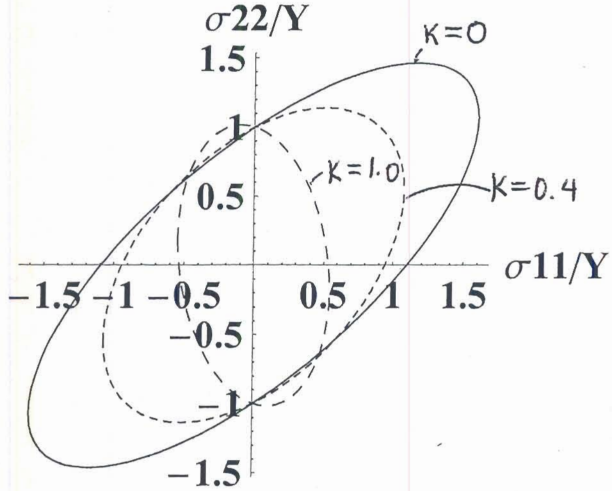
$$=\begin{bmatrix}\sigma_{11}-2K\sigma_{12}+K^2\sigma_{22} & \sigma_{12}-K\sigma_{22} & 0\\ \sigma_{12}-K\sigma_{22} & \sigma_{22} & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$[\pi']=\begin{bmatrix}\dfrac{2\sigma_{11}-4K\sigma_{12}+(2K^2-1)\sigma_{22}}{3} & \sigma_{12}-K\sigma_{22} & 0\\ \sigma_{12}-K\sigma_{22} & \dfrac{(2-K^2)\sigma_{22}-\sigma_{11}+2K\sigma_{12}}{3} & 0\\ 0 & 0 & \dfrac{-\sigma_{11}+2K\sigma_{12}-K^2\sigma_{22}}{3}\end{bmatrix}$$

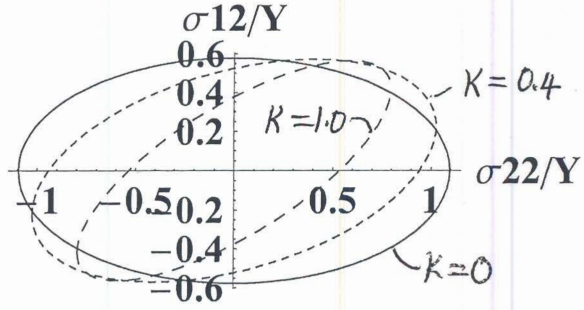
upon
Substitution, gets

$$\sigma_{11}^2-\left(\frac{4}{3}-2K^2\right)\sigma_{11}\sigma_{22}+\left(\frac{5}{6}+\frac{5K^2}{3}+K^4\right)\sigma_{22}^2+\left(3+4K^2\right)\sigma_{12}^2$$
$$-\left(4K\sigma_{11}+\frac{10}{3}K\sigma_{22}+4K^3\sigma_{22}\right)\sigma_{12}=Y^2$$

This equation may be compared to (11.192) which is the Mises yield criterion defined by the Cauchy stress.



$\sigma_{12} = 0$



$\sigma_{11} = 0$

Problem (10)

$$(\tau^{\langle 11\rangle}-\tau^{\langle 22\rangle})^2+(\tau^{\langle 22\rangle}-\tau^{\langle 33\rangle})^2+(\tau^{\langle 33\rangle}-\tau^{\langle 11\rangle})^2$$
$$+\,2(\tau^{\langle 12\rangle})^2+2(\tau^{\langle 23\rangle})^2+2(\tau^{\langle 31\rangle})^2 = f^2$$

Problem (11)

Simple shear $\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

By use of (11.194) and (11.196) the yield criterion of Problem (10) is expressed as

$$\frac{2}{1+K^2}\sigma_{11}^2 + \left(-2+\frac{4K^2}{1+K^2}\right)\sigma_{11}\sigma_{22} + 2\left(1+K^2+\frac{K^4}{1+K^2}\right)\sigma_{22}^2 + 2\left(1+\frac{4K^2}{1+K^2}\right)\sigma_{12}^2$$
$$-\frac{8K}{1+K^2}\left(\sigma_{11}+K^2\sigma_{22}\right)\sigma_{12} = f^2$$

Problem (12)

$$\left(\tilde{\sqrt{L}}^{11} - \tilde{\sqrt{L}}^{22}\right)^2 + \left(\tilde{\sqrt{L}}^{22} - \tilde{\sqrt{L}}^{33}\right)^2 + \left(\tilde{\sqrt{L}}^{33} - \tilde{\sqrt{L}}^{11}\right)^2$$
$$+ 2\left(\tilde{\sqrt{L}}^{12}\right)^2 + 2\left(\tilde{\sqrt{L}}^{23}\right)^2 + 2\left(\tilde{\sqrt{L}}^{31}\right)^2 = f^2 .$$

Simple shear: X_i are rectangular Cartesian coordinates

$\tilde{\sqrt{L}}^{ij} = \sqrt{L}^{ij}$ with $\underset{\sim}{E}_i = \underset{\sim}{e}_i$.

$$\underset{\sim}{F} = \begin{bmatrix} 1 & K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [\sqrt{L}] = \begin{bmatrix} \sigma_{11} - 2K\sigma_{12} + K^2\sigma_{22} & \sigma_{12} - K\sigma_{22} & 0 \\ \sigma_{12} - K\sigma_{22} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

upon substitution, the yield function is

$$2\left\{ \sigma_{11}^2 - (1-2K^2)\sigma_{11}\sigma_{22} + (1+K^4)\sigma_{22}^2 + (1+4K^2)\sigma_{12}^2 - 4K(\sigma_{11} + K^2\sigma_{22})\sigma_{12} \right\} = f^2$$

Problem (13)

$$2\phi = \left(\tilde{\xi}^{11}\right)^2 - (1+K^2)\,\tilde{\xi}^{11}\tilde{\xi}^{22} + (1+K^2)^2\left(\tilde{\xi}^{22}\right)^2 + 3(1+K^2)\left(\tilde{\xi}^{12}\right)^2$$
$$= Y^2(1+K^2), \quad \tilde{\xi}^{ij} = \tau^{ij} - \tilde{\alpha}^{ij} .$$

(11.200), $$\frac{D g^p_{ij}}{Dt} = \dot{\Lambda}\frac{\partial\phi}{\partial\tau^{ij}} = \dot{\Lambda}\frac{\partial\phi}{\partial\tilde{\xi}^{rs}}\frac{\partial\tilde{\xi}^{rs}}{\partial\tau^{ij}}$$

$$\frac{\partial\tilde{\xi}^{ij}}{\partial\tau^{rs}} = \delta^i_r\,\delta^j_s, \quad \frac{D g^p_{ij}}{Dt} = \dot{\Lambda}\frac{\partial\phi}{\partial\tilde{\xi}^{rs}}\delta^r_i\,\delta^s_j = \dot{\Lambda}\frac{\partial\phi}{\partial\tilde{\xi}^{ij}}$$

$$\frac{D g^p_{11}}{Dt} = \dot{\Lambda}\frac{\partial\phi}{\partial\tilde{\xi}^{11}} = \dot{\Lambda}\left[2\tilde{\xi}^{11} - (1+K^2)\tilde{\xi}^{22}\right]$$

$$\frac{D g^p_{22}}{Dt} = \dot{\Lambda}\left[-(1+K^2)\tilde{\xi}^{11} + 2(1+K^2)^2\tilde{\xi}^{22}\right] \quad \ldots\ldots (1)$$

$$\frac{D g^p_{12}}{Dt} = \dot{\Lambda}\; 6(1+K^2)\tilde{\xi}^{12}$$

(11.199), $\frac{1}{2}\frac{Dg^P_{ij}}{Dt} = \frac{\partial x_m}{\partial \theta^i}\frac{\partial x_n}{\partial \theta^j} D^P_{mn}$, $\theta^i = X_i$

use (11.189), we get

$$\frac{1}{2}\frac{Dg^P_{11}}{Dt} = D^P_{11}, \quad \frac{1}{2}\frac{Dg^P_{22}}{Dt} = K^2 D^P_{11} + 2K D^P_{12} + D^P_{22},$$

$$\frac{1}{2}\frac{Dg^P_{12}}{Dt} = K D^P_{11} + D^P_{12}.$$

or

$$D^P_{11} = \frac{1}{2}\frac{Dg^P_{11}}{Dt}, \quad D^P_{12} = \frac{1}{2}\frac{Dg^P_{12}}{Dt} - \frac{K}{2}\frac{Dg^P_{11}}{Dt},$$

$$D^P_{22} = \frac{1}{2}\frac{Dg^P_{22}}{Dt} + \frac{K^2}{2}\frac{Dg^P_{11}}{Dt} - K\frac{Dg^P_{12}}{Dt} \qquad \cdots (2)$$

Substituting (1) into (2), we obtain

$$D^P_{11} = \frac{1}{2}\dot{\Lambda}\left[2\tilde{\xi}^{11} - (1+k^2)\tilde{\xi}^{22}\right]$$

$$D^P_{12} = 3\dot{\Lambda}\,(1-K+K^2-K^3)\,\tilde{\xi}^{12} \qquad \cdots (3)$$

$$D^P_{22} = \dot{\Lambda}\left\{-\frac{1}{2}(1-K^2)\tilde{\xi}^{11} - 6K(1+K^2)\tilde{\xi}^{12} + \left(1+\frac{3K^2}{2}+\frac{K^4}{2}\right)\tilde{\xi}^{22}\right\}$$

We then use (11.196) and $\tilde{\alpha}^{ij} = \frac{\partial \theta^i}{\partial x_p}\frac{\partial \theta^j}{\partial x_q}\alpha_{pq}$, which leads to

$$\tilde{\alpha}^{11} = \alpha^{11} + K^2\alpha^{22} - 2K\alpha^{12}, \quad \tilde{\alpha}^{22} = \alpha^{22}, \quad \tilde{\alpha}^{12} = \alpha^{12} - K\alpha^{22}.$$

Eq. (3) reduces to

$$D^P_{11} = \dot{\Lambda}\left[\xi_{11} - 2K\xi_{12} - \frac{1}{2}(1-K^2)\xi_{22}\right]$$

$$D^P_{12} = 3(1-K)(1+K^2)(\xi_{12} - K\xi_{22}) \qquad \cdots (4)$$

$$D^P_{22} = -\dot{\Lambda}\left[\frac{1}{2}(1-K^2)\xi_{11} + K(5+7K)\xi_{12} - (1+7K^2+7K^4)\xi_{22}\right]$$

where $\xi_{ij} = \sigma_{ij} - \alpha_{ij}$.