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BOUNDARY INTEGRO-DIFFERENTIAL EQUATIONS OF ELLIPTIC BOUNDARY VALUE PROBLEMS AND THEIR NUMERICAL SOLUTIONS*

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ABSTRACT

In this paper, a new method of boundary reduction is proposed, which reduces the elliptic boundary value problems to integro-differential equations on the boundary and preserves the self-adjointness of the original problems. Moreover, a new boundary finite element method based on these integro-differential equations is presented and the error estimates of the numerical approximations are given. The numerical example shows that this new method is more effective.

Key words: integro-differential equation, boundary element method, numerical solution.

I. INTRODUCTION

In recent years, the boundary finite element method has been extensively applied to many fields of engineering and technology[1,2]. This method can be considered as a numerical method to obtain the numerical solution of the boundary value problem for partial differential equations by reducing them to the integral equations over the boundary. This method has an advantage of reducing the dimensionality of the problem by one dimension and hence produces a much smaller system of equations. But, in general, the self-adjointness of the original problem is not preserved after the reduction. From the computational point of view, the loss of the self-adjointness brings much trouble, such as more storage locations and more computer time. Among them there is an exception that the single layer theory can be applied to the Dirichlet problem of elliptic equations without loss of self-adjointness. In 1961, G. Fichera[3] succeeded in extending the single layer theory to the Dirichlet problem for strongly elliptic equations of higher order in two independent variables. The boundary integral equations deduced by Fichera's method preserve the self-adjointness of the original problems. After a little more than a decade, the variational formulations of these boundary integral equations amenable to finite element approximations were given. Furthermore the error estimates of the approximate solutions were obtained by J. C. Nedelec and J. Planchard[4], J. C. Nedelec[5], M. N. Le Roux[6] and G. C. Hsiao and W. L. Wendland[7]. Unfortu-

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nately, Fichera's method cannot be generalized to boundary conditions other than those of the Dirichlet type.

In 1978, Feng Kang[8] proposed the canonical boundary finite-element method by the canonical boundary reduction which faithfully preserves the essential characteristics of the original problems including the self-adjointness. In this direction, a series of results have been presented by Feng Kang and Yu Dehao[8-11]. In the procedure of canonical boundary reduction, the Green function of the original equation with the Dirichlet boundary condition is used, while the Green function could not be obtained for a general domain. Hence up to now the canonical boundary finite-element method for general domains has still been an open question.

In this paper, the double layer theory is applied to the Neumann problem of elliptic differential equations and elliptic boundary value problems are reduced to boundary integro-differential equations preserving the self-adjointness of original problems. On the basis of boundary integro-differential equations, a new boundary finite-element method is presented, which will be called I-D boundary finite-element method below. Besides, error estimates of I-D boundary finite-element approximations are also given.

Let Γ be a bounded simple closed curve in R^2. Γ is sufficiently smooth. Suppose Ω is the bounded domain with boundary Γ and Ω_c is the unbounded domain with boundary Γ. We consider the following Neumann problems of the Laplace equation and the Helmholtz equation:

$$\begin{cases} \Delta u = 0, \text{ in } \Omega, \\ \left.\dfrac{\partial u}{\partial n}\right|_\Gamma = g, \end{cases} \tag{1.1}$$

$$\begin{cases} \Delta u = 0, \text{ in } \Omega_c, \\ \left.\dfrac{\partial u}{\partial n^-}\right|_\Gamma = g, \qquad u \text{ is bounded, when } |x| \to +\infty, \end{cases} \tag{1.2}$$

$$\begin{cases} -\Delta u + u = 0, \text{ in } \Omega, \\ \left.\dfrac{\partial u}{\partial n}\right|_\Gamma = g, \end{cases} \tag{1.3}$$

$$\begin{cases} -\Delta u + u = 0, \text{ in } \Omega_c, \\ \left.\dfrac{\partial u}{\partial n^-}\right|_\Gamma = g, \qquad u \to 0, \text{ when } |x| \to +\infty, \end{cases} \tag{1.4}$$

where n and n^- denote the unit outward normal vectors on Γ for domains Ω and Ω_c respectively. $g \in H^{-\frac{1}{2}}(\Gamma)$ is a given function. As usual, $H^\alpha(\Gamma)$ and $H^m(\Omega)$ stand for Sobolev spaces, α and m are two real numbers. We know that there exists a unique week solution for the boundary value problem (1.3) or (1.4). For problems (1.1) and (1.2), the following condition is required

$$\int_\Gamma g(x)\, ds_x = 0. \tag{1.5}$$

Under the condition (1.5), problem (1.1) (or (1.2)) has a unique weak solution apart from a difference of a constant. A new method of boundary reduction will be presented in the following section.

II. A New Method of Boundary Reduction

To begin with, we consider the problem (1.1). Let

$$u(y)=\int_\Gamma \rho(x)\frac{\partial}{\partial n_x}\log|x-y|\,ds_x,\quad \forall y\in\Omega \tag{2.1}$$

be the solution of the problem (1.1), $\rho(x)$ being an undetermined function on Γ. $n_x=(n_x^1,\ n_x^2)$ refers to the unit outward normal vector at point $x\in\Gamma$. For every $y\in\Omega$ and an arbitrary unit vector $n_y=(n_y^1,\ n_y^2)$, we have

$$\frac{\partial u(y)}{\partial n_y}=\int_\Gamma \rho(x)\frac{\partial^2}{\partial n_y\,\partial n_x}\log|x-y|\,ds_x,\quad \forall y\in\Omega. \tag{2.2}$$

The crucial point of this new method of boundary reduction is the following lemma.

Lemma 2.1. *For every $x\neq y$, the following equality holds*

$$\frac{\partial^2}{\partial n_y\,\partial n_x}\log|x-y|=-\frac{\partial^2}{\partial \tau_y\,\partial \tau_x}\log|x-y|, \tag{2.3}$$

where $\tau_y=(-n_y^2,\ n_y^1)$ is a unit vector perpendicular to n_y, and $\tau_x=(-n_x^2,\ n_x^1)$ represents the unit tangent vector at point $x\in\Gamma$.

Proof. Let $x=(x_1,\ x_2)$, $y=(y_1,\ y_2)$, $r=|x-y|$,

$$H=\log|x-y|=\frac{1}{2}\log\{(x_1-y_1)^2+(x_2-y_2)^2\}.$$

Some computations yield

$$\frac{\partial^2 H}{\partial y_1\partial x_1}=\frac{(x_1-y_1)^2-(x_2-y_2)^2}{r^4},$$

$$\frac{\partial^2 H}{\partial y_2\partial x_1}=\frac{\partial^2 H}{\partial y_1\partial x_2}=\frac{2(x_1-y_1)(x_2-y_2)}{r^4},$$

$$\frac{\partial^2 H}{\partial y_2\partial x_2}=\frac{-(x_1-y_1)^2+(x_2-y_2)^2}{r^4};$$

then we have

$$\frac{\partial^2 H}{\partial n_y\,\partial n_x}=(n_y^1\ \ n_y^2)\begin{pmatrix}\frac{\partial^2 H}{\partial y_1\partial x_1} & \frac{\partial^2 H}{\partial y_1\partial x_2}\\ \frac{\partial^2 H}{\partial y_2\partial x_1} & \frac{\partial^2 H}{\partial y_2\partial x_2}\end{pmatrix}\begin{pmatrix}n_x^1\\ n_x^2\end{pmatrix}$$

$$=-(-n_y^2\ \ n_y^1)\begin{pmatrix}\frac{\partial^2 H}{\partial y_1\partial x_1} & \frac{\partial^2 H}{\partial y_1\partial x_2}\\ \frac{\partial^2 H}{\partial y_2\partial x_1} & \frac{\partial^2 H}{\partial y_2\partial x_2}\end{pmatrix}\begin{pmatrix}-n_x^2\\ n_x^1\end{pmatrix}$$

$$= -\frac{\partial^2 H}{\partial \tau_y \partial \tau_x}.$$

Q. E. D

Substituting (2.3) into (2.2) and using integration by parts, we obtain

$$\frac{\partial u(y_0)}{\partial n_{y_0}} = \frac{\partial}{\partial \tau_{y_0}} \int_\Gamma \rho'(x) \log |x - y_0| ds_x, \quad \forall y_0 \in \Omega, \tag{2.4}$$

where $\rho'(x) = \frac{d\rho(x)}{dS_x}$ and S_x is the arc-length along the boundary Γ. For every $y \in \Gamma$, we take $n_{y_0} = n_y$, n_y indicating the outward unit normal vector at point $y \in \Gamma$ and let y_0 go to y. Then we obtain

$$\frac{\partial u(y)}{\partial n_y} = \frac{d}{dS_y} \int_\Gamma \rho'(x) \log |x - y| dS_x, \quad \forall y \in \Gamma. \tag{2.5}$$

From the boundary condition of the problem (1.1), we obtain the following boundary integro-differential equation, which preserves the self-adjointness of the original problem:

$$\frac{d}{dS_y} \int_\Gamma \rho'(x) \log |x - y| dS_x = g(y), \quad \forall y \in \Gamma. \tag{2.6}$$

Similarly, let

$$u(y) = \int_\Gamma \rho(x) \frac{\partial}{\partial n_x} \log |x - y| dS_x, \quad \forall y \in \Omega \tag{2.7}$$

be the solution of the problem (1.2), where $\rho(x)$ is an undetermined function on Γ. From Lemma 2.1, we obtain

$$\left.\frac{\partial u(y)}{\partial n^-}\right|_\Gamma = -\frac{d}{dS_y} \int_\Gamma \rho'(x) \log |x - y| dS_x. \tag{2.8}$$

Moreover, the problem (1.2) can be reduced to the following boundary integro-differential equation:

$$\frac{d}{dS_y} \int_\Gamma \rho'(x) \log |x - y| dS_x = -g(y), \quad \forall y \in \Gamma. \tag{2.9}$$

We now proceed to discuss the problems (1.3) and (1.4). The fundamental solution of Eq. $-\Delta u + u = 0$ is the modified Bessel function of zero order $K_0(|x - y|)$. There is the expansion nearby $r = 0$

$$K_0(r) = \sum_{n=0}^{\infty} a_n r^{2n} \log \frac{1}{r} + \sum_{n=1}^{\infty} b_n r^{2n}, \quad a_0 = 1.$$

At infinity, we have

$$K_0(r) = \sqrt{\frac{\pi}{2r}}\, e^{-r} + \cdots$$

and $\lim_{r \to +\infty} K_0(r) = 0$. $K_0(r)$ satisfies the following differential equation

$$\frac{d^2K_0(r)}{dr^2} + \frac{1}{r}\frac{dK_0(r)}{dr} - K_0(r) = 0, \quad r \neq 0. \tag{2.10}$$

Let

$$u_1(y) = \int_\Gamma \rho(x)\frac{\partial}{\partial n_x} K_0(|x-y|)dS_x, \quad \forall y \in \Omega \tag{2.11}$$

be the solution of the problem (1.3), where function $\rho(x)$ will be determined below. For every $y \in \Omega$ and the unit vector $n_y = (n_y^1, n_y^2)$, we have

$$\frac{\partial u_1(y)}{\partial n_y} = \int_\Gamma \rho(x)\frac{\partial^2}{\partial n_y \partial n_x} K_0(|x-y|)dS_x, \quad \forall y \in \Omega. \tag{2.12}$$

For the derivative $\frac{\partial^2}{\partial n_y \partial n_x} K_0(|x-y|)$, we have

Lemma 2.2. *The following equality holds*

$$\frac{\partial^2 K_0(|x-y|)}{\partial n_y \partial n_x} = -\frac{\partial^2 K_0(|x-y|)}{\partial \tau_y \partial \tau_x} - K_0(|x-y|)\cos(n_x, n_y),$$
$$\forall x \neq y, \tag{2.13}$$

where $\tau_x = (-n_x^2, n_x^1)$ *and* $\tau_y = (-n_y^2, n_y^1)$.

Proof. Some computations yield

$$\frac{\partial^2 K_0(|x-y|)}{\partial y_1 \partial x_1} = -\frac{(x_1-y_1)^2}{r^2}\frac{d^2K_0(r)}{dr^2} - \frac{(x_2-y_2)^2}{r^3}\frac{dK_0(r)}{dr},$$

$$\frac{\partial^2 K_0(|x-y|)}{\partial y_2 \partial x_1} = \frac{\partial^2 K_0(|x-y|)}{\partial y_1 \partial x_2} = -\frac{(x_1-y_1)(x_2-y_2)}{r^2}\frac{d^2K_0(r)}{dr^2} + \frac{(x_1-y_1)(x_2-y_2)}{r^3}\frac{dK_0(r)}{dr},$$

$$\frac{\partial^2 K_0(|x-y|)}{\partial y_2 \partial x_2} = -\frac{(x_2-y_2)^2}{r^2}\frac{d^2K_0(r)}{dr^2} - \frac{(x_1-y_1)^2}{r^3}\frac{dK_0(r)}{dr}.$$

Eq. (2.10) leads to

$$\frac{\partial^2 K_0(|x-y|)}{\partial y_1 \partial x_1} + \frac{\partial^2 K_0(|x-y|)}{\partial y_2 \partial x_2} = -K_0(|x-y|). \tag{2.14}$$

On the other hand, we have

$$\frac{\partial^2 K_0(|x-y|)}{\partial n_y \partial n_x} = \frac{\partial^2 K_0(|x-y|)}{\partial y_1 \partial x_1} n_x^1 n_y^1 + \frac{\partial^2 K_0(|x-y|)}{\partial y_2 \partial x_1} n_x^1 n_y^2 + \frac{\partial^2 K_0(|x-y|)}{\partial y_1 \partial x_2} n_x^2 n_y^1 + \frac{\partial^2 K_0(|x-y|)}{\partial y_2 \partial x_2} n_x^2 n_y^2, \tag{2.15}$$

$$\frac{\partial^2 K_0(|x-y|)}{\partial \tau_y \partial \tau_x} = \frac{\partial^2 K_0(|x-y|)}{\partial y_1 \partial x_1} n_x^2 n_y^2 - \frac{\partial^2 K_0(|x-y|)}{\partial y_2 \partial x_1} n_x^2 n_y^1 - \frac{\partial^2 K_0(|x-y|)}{\partial y_1 \partial x_2} n_x^1 n_y^2 + \frac{\partial^2 K_0(|x-y|)}{\partial y_2 \partial x_2} n_x^1 n_y^1. \tag{2.16}$$

Combining (2.15), (2.16) and (2.14), we obtain

$$\frac{\partial^2 K_0(|x-y|)}{\partial n_y \partial n_x} + \frac{\partial^2 K_0(|x-y|)}{\partial \tau_y \partial \tau_x} = -K_0(|x-y|)(n_x^1 n_y^1 + n_x^2 n_y^2)$$
$$= -K_0(|x-y|)\cos(n_x, n_y).$$

This is the conclusion of Lemma 2.2.

Q. E. D.

Substituting (2.13) into (2.12) and integrating by parts, we obtain

$$\frac{\partial u_1(y_0)}{\partial n_{y_0}} = \frac{\partial}{\partial \tau_{y_0}} \int_\Gamma \rho'(x) K_0(|x-y_0|) dS_x$$
$$- \int_\Gamma \rho(x) K_0(|x-y_0|) \cos(n_x, n_{y_0}) dS_x, \tag{2.17}$$
$$\forall y_0 \in \Omega \text{ and unit vector } n_{y_0} = (n_{y_0}^1, n_{y_0}^2).$$

For every $y \in \Gamma$, n_y denotes the outward unit normal vector at point y. If $n_{y_0} = n_y$ and y_0 goes to y in (2.17), then we have

$$\frac{\partial u_1(y)}{\partial n_y}\Big|_\Gamma = \frac{d}{dS_y} \int_\Gamma \rho'(x) K_0(|x-y|) dS_x$$
$$- \int_\Gamma \rho(x) K_0(|x-y|) \cos(n_x, n_y) dS_x, \quad \forall y \in \Gamma. \tag{2.18}$$

By the boundary condition in the problem (1.3), we acquire the following integro-differential equation to determine function $\rho(x)$:

$$\frac{d}{dS_y} \int_\Gamma \rho'(x) K_0(|x-y|) dS_x - \int_\Gamma \rho(x) K_0(|x-y|) \cos(n_x, n_y) dS_x$$
$$= g(y), \quad \forall y \in \Gamma. \tag{2.19}$$

Similarly, let

$$u_2(y) = \int_\Gamma \rho(x) \frac{\partial}{\partial n_x} K_0(|x-y|) dS_x, \quad \forall y \in \Omega_c \tag{2.20}$$

be the solution of the problem (1.4) with function $\rho(x)$ to be determined. It is easy to check that $u_2(y)$ goes to zero when $|y|$ goes to infinity. We find

$$\frac{\partial u_2(y)}{\partial n^-}\Big|_\Gamma = -\frac{d}{dS_y} \int_\Gamma \rho'(x) K_0(|x-y|) dS_x$$
$$+ \int_\Gamma \rho(x) K_0(|x-y|) \cos(n_x, n_y) dS_x, \quad \forall y \in \Gamma. \tag{2.21}$$

Hence the problem (1.4) is reduced to the following boundary integro-differential equation

$$\frac{d}{dS_y} \int_\Gamma \rho'(x) K_0(|x-y|) dS_x - \int_\Gamma \rho(x) K_0(|x-y|) \cos(n_x, n_y) dS_x$$
$$= -g(y), \quad \forall y \in \Gamma. \tag{2.22}$$

III. Boundary Integro-Differential Eq. (2.6)

For every $g \in H^{-\frac{1}{2}}(\Gamma)$ satisfying the condition (1.5), the problem (2.6) is equivalent to the following variational problem:

$$\begin{cases}\text{Find } \rho(x) \in H^{\frac{1}{2}}(\Gamma) \text{ such that} \\ a(\rho, \varphi) = (g, \varphi), \ \forall \varphi \in H^{\frac{1}{2}}(\Gamma),\end{cases} \tag{3.1}$$

where

$$a(\rho, \varphi) = -\int_\Gamma \int_\Gamma \rho'(x)\varphi'(y) \log|x-y| \, dS_x dS_y,$$

$$(g, \varphi) = \int_\Gamma g(y)\varphi(y) dS_y.$$

The bilinear form $a(\rho, \varphi)$ has been discussed in detail by G. C. Hsiao and W. L. Wendland[7]. The main results in [7] are the following

Lemma 3.1.

(i) *$a(\rho, \varphi)$ is a bounded bilinear form on $H^{\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$, namely, there exists a constant $M > 0$ such that*

$$|a(\rho, \varphi)| \leqslant M\|\rho'\|_{-\frac{1}{2},\Gamma}\|\varphi'\|_{-\frac{1}{2},\Gamma}, \ \forall \rho, \ \varphi \in H^{\frac{1}{2}}(\Gamma). \tag{3.2}$$

(ii) *Suppose that* diam $\Omega :=$ *the diameter of* $\Omega < 1$, *then there is a constant* $\nu > 0$ *such that*

$$a(\rho, \rho) \geqslant \nu \|\rho'\|^2_{-\frac{1}{2},\Gamma}, \ \forall \rho \in H^{\frac{1}{2}}(\Gamma). \tag{3.3}$$

Let

$$N_0 = \{\rho \in H^{\frac{1}{2}}(\Gamma), \ a(\rho, \rho) = 0\}.$$

By the inequality (3.3), we know that the dimensionality of null space N_0 is one. Obviously, $\rho \equiv 1 \in N_0$. From the Fredholm alternative theorem, we have

Theorem 3.1. (1.5) *is the necessary and sufficient condition for the existence of the solution of the variational problem* (3.1).

Now let us introduce space

$$V_0 = \left\{\rho \in H^{\frac{1}{2}}(\Gamma), \ \int_\Gamma \rho(x) dS_x = 0\right\},$$

V_0 being a subspace of $H^{\frac{1}{2}}(\Gamma)$. Moreover the norm $\|\rho\|_{\frac{1}{2},\Gamma}$ and the semi-norm $\|\rho'\|_{-\frac{1}{2},\Gamma}$ are equivalent in space V_0. Let $\|\rho\|_{V_0} = \|\rho'\|_{-\frac{1}{2},\Gamma}$. Consider the following variational problem:

$$\begin{cases}\text{Find } \rho \in V_0 \text{ such that} \\ a(\rho, \varphi) = (g, \varphi) \ \forall \varphi \in V_0.\end{cases} \tag{3.4}$$

By Lemma 3.1 and the Lax-Milgram Theorem, we have the following result.

Theorem 3.2. *The variational problem* (3.4) *has a unique solution* $\rho_0 \in V_0$, *and*

$$\|\rho_0\|_{V_0} \leqslant \frac{M}{\nu} \|g\|_{-\frac{1}{2},\Gamma}. \tag{3.5}$$

Under the condition (1.5), ρ_0 is a solution of the problem (3.1) and $\rho = \rho_0 + C$ are the solutions of the problem (3.1) for every constant C.

The remainder of this section is devoted to finite element approximation of the problem (3.4).

Suppose V_0^h is a finite dimensional subspace of V_0. Consider the following approximate problem:

$$\begin{cases}\text{Find } \rho_h \in V_0^h \text{ such that} \\ a(\rho_h, \varphi_h) = (g, \varphi_h), \ \forall \varphi_h \in V_0^h.\end{cases} \tag{3.6}$$

We propose

Theorem 3.3. *The variational problem* (3.6) *has a unique solution* ρ_h, *and the following abstract error estimate holds*

$$\|\rho_0 - \rho_h\|_{V_0} \leqslant \frac{M}{\nu} \inf_{\varphi_h \in V_0^h} \|\rho_0 - \varphi_h\|_{V_0}. \tag{3.7}$$

This conclusion follows immediately from the Lax-Milgram Theorem and Cea Lemma[12].

IV. Boundary Integro-Differential Eq. (2.19)

For every $g \in H^{-\frac{1}{2}}(\Gamma)$, the problem (2.19) is equivalent to the following variational problem:

$$\begin{cases}\text{Find } \rho(x) \in H^{\frac{1}{2}}(\Gamma) \text{ such that} \\ b(\rho, \varphi) = -(g, \varphi), \ \forall \varphi \in H^{\frac{1}{2}}(\Gamma),\end{cases} \tag{4.1}$$

where

$$\begin{aligned} b(\rho, \varphi) = & \int_\Gamma \int_\Gamma \rho'(x)\varphi'(y) K_0(|x-y|) dS_x dS_y \\ & + \int_\Gamma \int_\Gamma \rho(x)\varphi(y) K_0(|x-y|) \cos(n_x, n_y) dS_x dS_y. \end{aligned} \tag{4.2}$$

For bilinear form $b(\rho, \varphi)$, we have

Lemma 4.1. $b(\rho, \varphi)$ *is a bounded bilinear form on* $H^{\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$, *i.e. there is a constant* $M > 0$, *such that*

$$|b(\rho, \varphi)| \leqslant M \|\rho\|_{\frac{1}{2},\Gamma} \|\varphi\|_{\frac{1}{2},\Gamma}, \ \forall \rho, \ \varphi \in H^{\frac{1}{2}}(\Gamma). \tag{4.3}$$

The proof is similar to that of Lemma 3.1(i) and is omitted here.

To prove the coerciveness of $b(\rho, \varphi)$, we define the linear operator $\mathscr{K}$ from $H^{1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$ as follows:

$$\mathscr{K}\rho = \int_\Gamma \rho(x) \frac{\partial}{\partial n_x} K_0(|x-y|) dS_x - \pi\rho(y), \ \forall \rho \in H^{\frac{1}{2}}(\Gamma). \tag{4.4}$$

We know that $\frac{\partial}{\partial n_x} K_0(|x-y|)$ and $\frac{\partial}{\partial n_x}\log\frac{1}{|x-y|}$ have the same singularity on $\Gamma\times\Gamma$. Hence the integral operator in (4.4) is completely continuous and bounded. For every $v\in H^{1/2}(\Gamma)$, consider

$$v=\int_\Gamma \rho(x)\frac{\partial K_0(|x-y|)}{\partial n_x}dS_x-\pi\rho(y). \tag{4.5}$$

We know that the homogeneous integral equation

$$0=\int_\Gamma \rho(x)\frac{\partial K_0(|x-y|)}{\partial n_x}dS_x-\pi\rho(y)$$

has a unique solution $\rho\equiv 0$. By the Fredholm alternative theorem, we obtain that for every $v\in H^{1/2}(\Gamma)$, the problem (4.5) has a unique solution $\rho\in H^{1/2}(\Gamma)$. Hence $\mathscr{K}: H^{1/2}(\Gamma)\to H^{1/2}(\Gamma)$ is a bijective mapping and is continuous. Then the Banach Theorem implies the continuity of the inverse $\mathscr{K}^{-1}: H^{1/2}(\Gamma)\to H^{1/2}(\Gamma)$, which is equivalent to the following fact: There is a constant $\nu>0$ such that

$$\|\mathscr{K}\rho\|_{\frac{1}{2},\Gamma}\geqslant \nu\|\rho\|_{\frac{1}{2},\Gamma},\quad \forall\rho\in H^{\frac{1}{2}}(\Gamma). \tag{4.6}$$

Lemma 4.2. *There exists a constant $\mu>0$ such that*

$$b(\rho,\rho)\geqslant \mu\|\rho\|^2_{\frac{1}{2},\Gamma},\quad \forall\rho\in H^{\frac{1}{2}}(\Gamma). \tag{4.7}$$

Proof. Because space $C^\infty(\Gamma)$ is dense in $H^{\frac{1}{2}}(\Gamma)$, we only need to prove (4.7) for $\rho\in C^\infty(\Gamma)$. For every $\rho\in C^\infty(\Gamma)$, let

$$u_1(y)=\int_\Gamma \rho(x)\frac{\partial}{\partial n_x}K_0(|x-y|)dS_x\quad \forall y\in\Omega,$$

$$u_2(y)=\int_\Gamma \rho(x)\frac{\partial}{\partial n_x}K_0(|x-y|)dS_x\quad \forall y\in\Omega_c.$$

By the boundary behaviour of the double layers, we obtain

$$u_1(y)|_\Gamma=\int_\Gamma \rho(x)\frac{\partial}{\partial n_x}K_0(|x-y|)dS_x-\pi\rho(y)\equiv\mathscr{K}\rho,$$

$$u_2(y)|_\Gamma=\int_\Gamma \rho(x)\frac{\partial}{\partial n_x}K_0(|x-y|)dS_x+\pi\rho(y),$$

and

$$u_1(y)|_\Gamma-u_2(y)|_\Gamma=-2\pi\rho(y).$$

From (2.18) and (2.21), we know that

$$\begin{aligned}\frac{\partial u_1(y)}{\partial n_y}\Big|_\Gamma&=-\frac{\partial u_2(y)}{\partial n_y^-}\Big|_\Gamma\\&=\frac{d}{dS_y}\int_\Gamma \rho'(x)K_0(|x-y|)dS_x\\&\quad-\int_\Gamma \rho(x)K_0(|x-y|)\cos(n_x,n_y)dS_x.\end{aligned}$$

On the other hand, when $|y|\to+\infty$, we have

$$u_2(y) = o\left(\frac{1}{|y|^2}\right),$$

$$\frac{\partial u_2(y)}{\partial y_1} = o\left(\frac{1}{|y|^2}\right),$$

$$\frac{\partial u_2(y)}{\partial y_2} = o\left(\frac{1}{|y|^2}\right).$$

An application of the Green formulation yields

$$\iint_\Omega (\nabla u_1 \cdot \nabla u_1 + u_1^2)dy + \iint_{\Omega_c} (\nabla u_2 \cdot \nabla u_2 + u_2^2)dy$$

$$= \int_\Gamma \frac{\partial u_1}{\partial n_y} u_1 dS_y + \int_\Gamma \frac{\partial u_2}{\partial n_{\bar{y}}} u_2 dS_y$$

$$= \int_\Gamma \frac{\partial u_1}{\partial n_y}(u_1 - u_2)dS_y = 2\pi b(\rho, \rho).$$

Hence we have

$$b(\rho, \rho) \geqslant \frac{1}{2\pi}\|u_1\|_{1,\Omega}^2. \tag{4.8}$$

From the trace theorem, we know that there is a constant $C_1 > 0$ such that

$$\|u_1\|_{1,\Omega} \geqslant C_1\|u_1\|_{\frac{1}{2},\Gamma} = C_1\|\mathscr{K}\rho\|_{\frac{1}{2},\Gamma}. \tag{4.9}$$

Combining the inequalities (4.8), (4.9) and (4.6), we admit that the inequality (4.7) will immediately follow with $\mu = \dfrac{C_1^2\nu^2}{2\pi}$.

An application of the Lax-Milgram Theorem yields

Theorem 4.1. *For every* $g \in H^{-1/2}(\Gamma)$, *the variational problem* (4.1) *has a unique solution* $\rho \in H^{1/2}(\Gamma)$, *and*

$$\|\rho\|_{\frac{1}{2},\Gamma} \leqslant \frac{M}{\mu}\|g\|_{-\frac{1}{2},\Gamma}. \tag{4.10}$$

Now suppose V^h is a finite dimensional subspace of $H^{1/2}(\Gamma)$. Consider the following approximate problem

$$\begin{cases} \text{Find } \rho_h \in V^h \text{ such that} \\ b(\rho_h, \varphi_h) = -(g, \varphi_h), \ \forall \varphi_h \in V^h. \end{cases} \tag{4.11}$$

The following theorem arises.

Theorem 4.2. *The variational problem* (4.11) *has a unique solution* $\rho_h \in V^h$, *and the following abstract error estimate holds*

$$\|\rho - \rho_h\|_{\frac{1}{2},\Gamma} \leqslant \frac{M}{\mu} \inf_{\varphi_h \in V^h} \|\rho - \varphi_h\|_{\frac{1}{2},\Gamma}, \tag{4.12}$$

where ρ *is the solution of the problem* (4.1).

This conclusion follows from the Lax-Milgram Theorem and the Cea Lemma.

V. Numerical Example

Consider the following example:

$$\begin{cases} \Delta u = 0, \text{ in } \Omega, \\ \left.\dfrac{\partial u}{\partial n}\right|_{\Gamma} = g, \end{cases} \tag{5.1}$$

where

$$\Omega = \left\{ (x_1, x_2) \in \mathbb{R}^2 \,\middle|\, \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} < 1,\ a = \frac{1}{2},\ b = \frac{1}{4} \right\}$$

is an ellipse, the parametric equation of the boundary Γ is

$$\begin{cases} x_1 = a\cos t \\ x_2 = b\sin t \end{cases} \qquad (0 < t \leqslant 2\pi) \tag{5.2}$$

and

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma} = g = \frac{ab\cos 2t}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}.$$

First of all, the boundary Γ is divided into eight segmental arcs by eight nodes $a_4, a_8, a_{12}, a_{16}, a_{20}, a_{24}, a_{28}, a_{32}$ as shown in Fig. 1. The division is denoted by partition I. Then the partition is refined by dividing every segmental arcs into two parts. We obtain the partition II consisting of 16 segmental arcs corresponding to the nodes $\{a_i, i = 2,4,6,8,\cdots,32\}$. Refine it again, and then the final partition III consists of 32 segmental arcs as shown in Fig. 1.

u_{I}, u_{II} and u_{III} denote the I-D boundary finite-element approximations of the problem (5.1) corresponding to the partitions I, II and III and piecewise linear boundary elements. u denotes the exact solution of the problem (5.1). We get the values of $u_{\text{I}}, u_{\text{II}}, u_{\text{III}}$ and u on the nodes $\{a_i, i = 2,4,6,\cdots,32\}$.

The relative errors $\dfrac{|u - u_K|}{\max\limits_{1\leqslant i\leqslant 32} |u(a_i)|}$ $(K = \text{I},\text{II},\text{III})$ are given in Fig. 2. This nu-

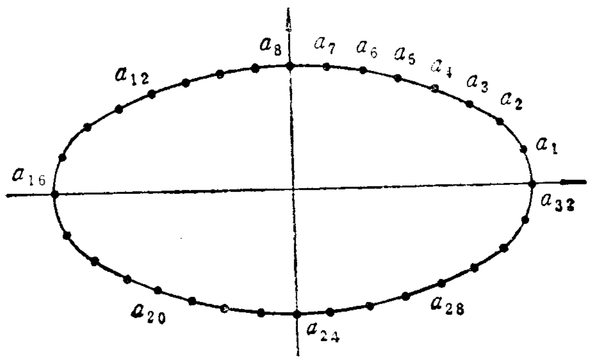

Fig. 1. Partitions I—III.

Table 1

Point	2	4	6	8	10	12
u_{I}	$.81715\times10^{-1}$	$.35171\times10^{-1}$	-0.10652×10^{-1}	-0.53024×10^{-1}	-0.91464×10^{-2}	$.37701\times10^{-1}$
u_{II}	$.10140$	$.44413\times10^{-1}$	-0.12611×10^{-1}	-0.36246×10^{-1}	-0.12599×10^{-1}	$.44493\times10^{-1}$
u_{III}	$.10214$	$.46949\times10^{-1}$	-0.82456×10^{-2}	-0.31109×10^{-1}	-0.82459×10^{-2}	$.46949\times10^{-1}$
u	$.10212$	$.46875\times10^{-1}$	-0.83677×10^{-2}	-0.31250×10^{-1}	-0.83677×10^{-2}	$.46875\times10^{-1}$

point	14	16	18	20	22	24
u_{I}	$.84529\times10^{-1}$	$.12779$	$.84529\times10^{-1}$	$.37701\times10^{-1}$	-0.91464×10^{-2}	-0.53024×10^{-1}
u_{II}	$.10159$	$.12524$	$.10159$	$.44493\times10^{-1}$	-0.12600×10^{-1}	-0.36246×10^{-1}
u_{III}	$.10215$	$.12501$	$.10215$	$.46949\times10^{-1}$	-0.82465×10^{-2}	-0.31108×10^{-1}
u	$.10212$	$.12500$	$.10212$	$.46875\times10^{-1}$	-0.83677×10^{-2}	-0.31250×10^{-1}

Point	26	28	30	32
u_{I}	-0.10652×10^{-1}	$.35171\times10^{-1}$	$.81716\times10^{-1}$	$.12500$
u_{II}	-0.12611×10^{-1}	$.44413\times10^{-1}$	$.10140$	$.12500$
u_{III}	-0.82453×10^{-2}	$.46949\times10^{-1}$	$.10214$	$.12500$
u	-0.83677×10^{-2}	$.46875\times10^{-1}$	$.10212$	$.12500$

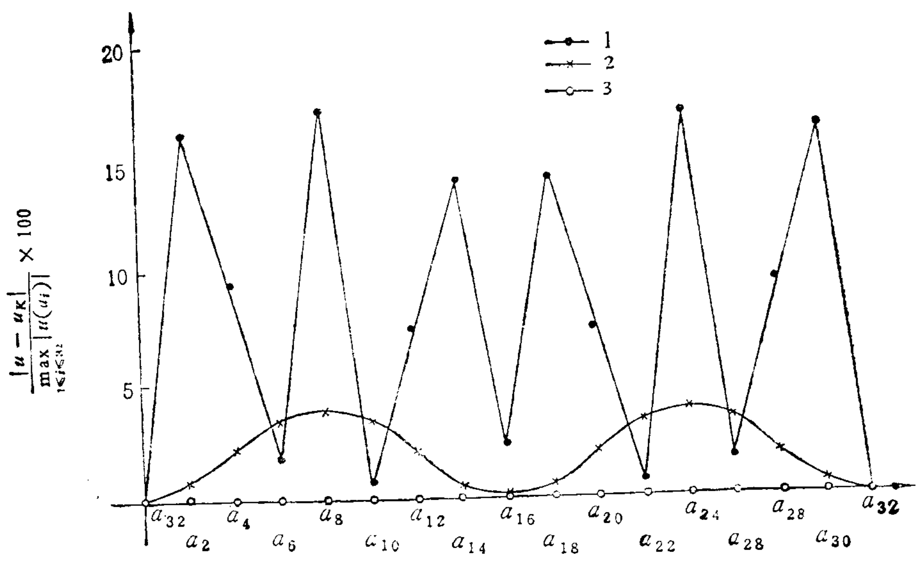

Fig. 2 Relative errors.

merical example shows the I-D boundary finite-element method is very effective.

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