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Application of symmetric indirect Trefftz method to free vibration problems in 2D

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SUMMARY

A symmetric indirect Trefftz method is developed to solve the free vibration problem of a 2D membrane. It is proved that in this approach the spurious eigensolution exists, and an auxiliary matrix is constructed to help extraction of the spurious solution using the generalized singular-value decomposition. In addition to the spurious eigensolution, this regular formulation suffers from its ill-posed nature, i.e. the numerical instability. In order to deal with the numerical instability, the Tikhonov's regularization method, in conjunction with the generalized singular-value decomposition, is suggested. The proposed approach has some merits when compared with other regular boundary element formulations reported so far; namely the capacity of representing eigenmodes and the ability to deal with a multiply connected domain of genus 1. Several numerical examples are demonstrated to show the validity of the current approach. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: Trefftz method; regular boundary element; generalized singular-value decomposition; Tikhonov's regularization method

1. INTRODUCTION

The eigenproblems are extremely important in many fields of engineering. It is well known that the engineer should avoid designing the eigenfrequencies of the structure to coincide with the driving force frequency. In addition, the eigenvalues and corresponding eigenfunctions are both used to represent arbitrary functions in the linear theory of vibration analysis, which means that they construct the spectrum of the operator [1]. It is then not surprising that eigenproblem analysis becomes the first step in exploring the wonderful world of vibration problems. For an arbitrarily shaped domain, the numerical methods are usually required in analysis since the analytical solution might not be available. Among them, the finite element method (FEM) and

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the boundary element method (BEM) are more attractive to both academic and engineering fields due to their merits of own numerical calculations. The main advantage of the BEM as applied to vibration problems in time or frequency domain formulations is the dimensionality reduction and this has led to the proposals of various BEM formulation schemes [2–4]. For a Helmholtz equation, the complex-valued singular boundary elements have been employed by Kamiya *et al.* [3] in solving the eigenproblems. To avoid complicated computation in the domain of a complex number, two alternatives, the real-part formulation and regular formulation were proposed by De Mey [5]. The real-part formulation basically adopts the real-part function of the complex-valued auxiliary function (the fundamental solution) as the auxiliary function. The multiple reciprocity boundary element method (MR/BEM), which treats the Helmholtz equation as a Poisson's equation with an external source, has been developed to transform the domain integral into boundary integrals [6] and applied to solve the eigenproblem [7]. Basically the MR/BEM uses a real-valued computation. The relationship between the real-part formulation suggested by De Mey and the MR/BEM was not clear until Yeih's work [7]. Yeih *et al.* [7] proved that the real-part formulation and the MR/BEM are equivalent mathematically, and the spurious eigenvalues encountered in both formulations stem from lacking constraint equations contributed by the imaginary part of the complex-valued fundamental solution. Many efforts were reported on treating the spurious eigensolution for solving the eigenproblem using the real-part formulation or the multiple reciprocity method [8, 9]. Another approach proposed by De Mey is the regular formulation. This method adopts a non-singular auxiliary function to construct the constraint equations. Kim and Kang [2] used the wave-type base functions, one regular formulation in our opinion, to analyse the free vibration of membranes. In their paper, the wave-type base functions, which are periodic along each element and propagating into the domain of interest, were selected to construct the needed equations. They pointed out that some incorrect answers would appear and they explained this phenomenon as the incompleteness of the basis functions. Later, Kang *et al.* [4, 10] proposed another regular formulation using the so-called non-dimensional dynamic influence function. Simply speaking, their method took the response at any point inside the domain of interest as a linear combination of many non-singular point sources located on the selected boundary nodes. They claimed that their method worked very well and no numerical instability behaviours were reported, which was later criticized by Chen *et al.* [11]. Kang's method is an indirect method such that it can represent mode shape easily. Recently, Chen *et al.* [12] used the circular domain and the property of circulants to examine theoretically the possibility of using the imaginary dual BEM as a solver for the Helmholtz eigenproblems. They reported that spurious eigensolutions also appeared in the imaginary dual BEM; however, no numerical examples were illustrated in their paper. Kuo *et al.* [13] pointed out that the ill-posed behaviour should exist in the regular BEM formulation and they also proposed a combination of the Tikhonov's regularization method and the generalized singular-value decomposition to treat such an ill-posed formulation. The regular formulations Kuo *et al.* proposed are a combination of the imaginary-part direct dual BEM and the plane wave method. In their paper, the mathematical structure of using the regular boundary integral formulation to solve the free vibration problem was explained very clearly. However, their proposed methods although can deal with the numerical instability but have two unfavourable properties: first, their methods cannot represent the mode shape because they are direct type regular boundary formulations; second, their methods fail in treating a multiply connected domain. Such a drawback although can be overcome by introducing an artificial boundary as suggested in Reference [10], it loses the merit of boundary element method, i.e. discretization on boundary solely, and it requires users to construct an artificial boundary.

Another regular boundary type approach is the Trefftz method, which has been widely used to deal with many types of problems, such as plane elasticity problems [14], plate bending problems [15] and acoustics [16], and the first applications of various modern forms of this method can be traced back to 1978 [17]. The boundary type Trefftz method basically employs the complete set of solutions satisfying the governing equation as the beginning step. To derive the boundary integral equation, either the reciprocity law, which is similar to those used in the conventional BEMs, or the weight residual method can be used. A main benefit for the Trefftz method is that it does not involve singular integrals due to the properties of its solution basis functions (T functions); thus, it can be categorized into the regular boundary element method. Besides, this advocated approach yields a solution that offers simultaneously the advantages of the classical FEM and BEM solutions, without having their drawbacks [17]. A thorough review article about the Trefftz method can be found in Reference [17]. In addition, some successful applications of special purposes functions, such as those for a circular hole, elliptical holes and obtuse or reentrant corners are also reported in Reference [17]. Although the Trefftz method has been successfully used in solving many problems, to the eigenproblem using the Helmholtz equation few attempts [18] have been found in the literature to authors' best knowledge. The reason may come from the ill-posed behaviour nature of a regular formulation as Kuo *et al.* [13] mentioned.

In this paper, our purpose is to construct an indirect Trefftz method to solve the free vibration problem and satisfy the following requirements:

(1) the proposed method should be able to deal with the spurious eigensolutions if they exist;
(2) the proposed method can overcome the numerical instability of regular boundary formulations;
(3) the proposed method can deal with a multiply connected domain without introducing any artificial boundary;
(4) the proposed method can represent the eigenmode within its own formulation.

2. FORMULATIONS

2.1. Problem set-up and the difficulty in the conventional indirect Trefftz method

Consider a 2D membrane Ω enclosed by the boundary Γ, the governing equation of free vibration of a membrane can be modeled by the Helmholtz equation as

$$(\nabla^2 + k^2)u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega \tag{1}$$

where ∇^2 is the Laplace operator, k is the wave number, and $u(\mathbf{x})$ is the physical quantity at $\mathbf{x}$.

The direct Trefftz method is constructed as follows. Let a field $W(\mathbf{x})$ satisfy the Helmholtz equation, i.e.

$$(\nabla^2 + k^2)W(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega \tag{2}$$

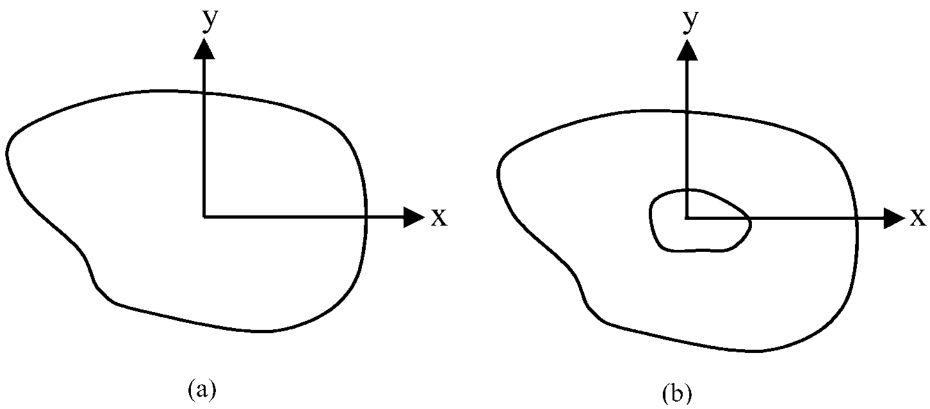


Figure 1. (a) A simply connected domain; and (b) a multiply connected domain of genus 1.

then by the reciprocity theorem one can have

$$\int_{\Gamma} W(\mathbf{x}) \frac{\partial u(\mathbf{x})}{\partial n} \, \mathrm{d}\Gamma(\mathbf{x}) = \int_{\Gamma} u(\mathbf{x}) \frac{\partial W(\mathbf{x})}{\partial n} \, \mathrm{d}\Gamma(\mathbf{x}) \tag{3}$$

where n denotes the outnormal direction at the boundary point $\mathbf{x}$. The choice of $W(\mathbf{x})$ depends on the problem itself. A complete set of $W(\mathbf{x})$, written as $\{W_i(\mathbf{x})\}$, is chosen to give enough bases to represent physical quantities. This complete set is called the T-complete function set. In the mathematical language, the T-complete function set provides complete function bases to represent any physical fields. For example, for a simply connected domain shown in Figure 1(a) and having the origin located inside the interested domain, it is convenient to have the T-complete set as

$$\{J_0(kr), J_m(kr)\cos(m\theta), J_m(kr)\sin(m\theta)\} \quad \text{for } m = 1, 2, 3 \ldots$$

in which J_m is the Bessel function of mth order, r is the distance from the origin to a domain point and θ is the angle between the x-axis and the radial vector from the origin to that domain point. For a multiply connected domain of genus 1 (i.e. with one hole) and locating the origin inside the hole as shown in Figure 1(b), the T-complete set is

$$\{J_0(kr), Y_0(kr), J_m(kr)\cos(m\theta), J_m(kr)\sin(m\theta), Y_m(kr)\cos(m\theta), Y_m(kr)\sin(m\theta)\}$$

for $m = 1, 2, 3, \ldots$, where Y_m is the second kind Bessel function of mth order.

To derive the indirect Trefftz method, an intuitive way is to derive an indirect method using the direct one, i.e. using Equation (3). The spirit of the indirect method is to represent physical quantities by a linear combination of bases, that is

$$u(\mathbf{x}) = \sum_q a_q W_q(\mathbf{x}) \tag{4a}$$

$$t(\mathbf{x}) = \frac{\partial u(\mathbf{x})}{\partial n} = \sum_q a_q \frac{\partial W_q(\mathbf{x})}{\partial n} \tag{4b}$$

where $W_q(\mathbf{x})$ is the qth base function used in the Trefftz method, and a_q is the undetermined coefficient. Substituting (4a) and (4b) into (3), we have

$$\int_\Gamma \sum_q W_i(\mathbf{x}) \frac{\partial W_q(\mathbf{x})}{\partial n} a_q \, \mathrm{d}\Gamma(\mathbf{x}) = \int_\Gamma \sum_q \frac{\partial W_i(\mathbf{x})}{\partial n} W_q(\mathbf{x}) a_q \, \mathrm{d}\Gamma(\mathbf{x}) \tag{5}$$

Equation (5) is a conventional indirect Trefftz method. In the following, we will examine the difficulty of using (5) to solve the free vibration problem.

Let us consider a Dirichlet problem, $u=0$, on the boundary of a circular domain with the radius equal to 1, (5) becomes

$$\int_\Gamma \sum_q W_i(\mathbf{x}) \frac{\partial W_q(\mathbf{x})}{\partial n} a_q \, \mathrm{d}\Gamma(\mathbf{x}) = 0 \tag{6}$$

It is easy to prove that (6) has a spurious eigensolution resulting from $W_i(\mathbf{x})$, i.e. from the weighting function. It is not our purpose to use (6), therefore, we skip the proof. For details of the derivation, interesting readers can refer to Kuo's work [13]. The method to prove is basically the same as using the property of a circulant matrix.

Also while solving (6) we will encounter the ill-posed behaviour as the element number increases. Therefore, we need to construct the auxiliary problem according to Kuo's suggestion [13]. Let us take the Neumann problem as the auxiliary problem for example, we have

$$\int_\Gamma \sum_q \frac{\partial W_i(\mathbf{x})}{\partial n} W_q(\mathbf{x}) a_q \, \mathrm{d}\Gamma(\mathbf{x}) = 0 \tag{7}$$

It then can be seen that the leading matrices in (6) and (7) are transpose to each other. Remember that the original idea for doing so is to extract the spurious eigenvalue out from both systems. However, the true eigensolution of the original problem in (6) is the same as the spurious eigensolution in (7) and *vice versa*. It makes the algorithm suggested by Kuo *et al*. fail. Furthermore, this method only can deal with simple boundary conditions such as the Dirichlet or Neumann problem but cannot deal with a general boundary condition, $\alpha_1 u + \beta_1 t = 0$.

2.2. *Symmetric indirect Trefftz method*

Now let us try to construct a symmetric indirect Trefftz method for solving a general boundary value problem, $\alpha_1 u + \beta_1 t = 0$. First, we propose to replace the boundary conditions by

$$\alpha_1 u + \beta_1 t = \bar{g} \tag{8}$$

then select the weighted function as $\beta_1 W_i - \alpha_1(\partial W_i/\partial n)$ and a boundary integral equation can be formulated as

$$\int_\Gamma \left(\beta_1 W_i - \alpha_1 \frac{\partial W_i}{\partial n} \right) (\alpha_1 u + \beta_1 t - \bar{g}) \, \mathrm{d}\Gamma(\mathbf{x}) = 0 \tag{9}$$

Now let us substitute (4a) and (4b) into (9), we have

$$[K_{ij}][a_j] = [H_{im}][\bar{g}_m] \tag{10}$$

where

$$K_{ij} = \int_\Gamma \left(\beta_1 W_i - \alpha_1 \frac{\partial W_i}{\partial n}\right)\left(\alpha_1 W_j + \beta_1 \frac{\partial W_j}{\partial n}\right) d\Gamma(\mathbf{x})$$
$$= \int_\Gamma \alpha_1 \beta_1 \left[W_i(\mathbf{x}) W_j(\mathbf{x}) - \frac{\partial W_i(\mathbf{x})}{\partial n}\frac{\partial W_j(\mathbf{x})}{\partial n}\right] d\Gamma(\mathbf{x})$$
$$+ \int_\Gamma \left[\beta_1^2 W_i(\mathbf{x}) \frac{\partial W_j(\mathbf{x})}{\partial n} - \alpha_1^2 \frac{\partial W_i(\mathbf{x})}{\partial n} W_j(\mathbf{x})\right] d\Gamma(\mathbf{x}) \quad (11a)$$

and

$$H_{im} = \int_{\Gamma_m} \left(\beta_1 W_i(\mathbf{x}) - \alpha_1 \frac{\partial W_i}{\partial n}\right) d\Gamma(\mathbf{x}) \quad (11b)$$

in which Γ_m is the mth element on the boundary.

In the formulation of (11a), the matrix $[K_{ij}]$ is a symmetric one. The proof is shown in the following. Considering that

$$K_{ij} - K_{ji} = \int_\Gamma \beta_1^2 \left(W_i \frac{\partial W_j}{\partial n} - W_j \frac{\partial W_i}{\partial n}\right) d\Gamma - \int_\Gamma \alpha_1^2 \left(\frac{\partial W_i}{\partial n} W_j - W_i \frac{\partial W_j}{\partial n}\right) d\Gamma \quad (12)$$

then, (12) leads to zero since the reciprocity theorem tells us that

$$\int_\Gamma \left(W_i \frac{\partial W_j}{\partial n} - W_j \frac{\partial W_i}{\partial n}\right) d\Gamma = 0 \quad (13)$$

The auxiliary matrix, [**H**] in (11b), is constructed in order to help extraction of spurious eigensolutions. This matrix is not necessary to be a square matrix and is not a symmetric one even when it is a square one. To check the rank deficiency of [**K**] and [**H**] matrices, we will have an overall description as follows.

First, let us confirm that the true eigensolution will cause the rank deficiency of the [**K**] matrix but will not cause the rank deficiency of the [**H**] matrix. Suppose that the true eigensolution, u, can be written as (4a) and satisfies the boundary condition, $\alpha_1 u + \beta_1 t = 0$, then we will have

$$[K_{ij}][a_j] = 0 \quad (14)$$

for a non-trivial [**a**]. This means that the true eigensolution makes the matrix [**K**] degenerated. This result comes directly from substituting (4a) and (4b) into the first line in (11a). The true eigensolution itself cannot result in the rank deficiency of the [**H**] matrix on the other hand. Considering the following equation:

$$[a_i]^{\mathrm{T}}[H_{im}] = \int_{\Gamma_m} \left[\sum_i \left(\beta_1 W_i - \alpha_1 \frac{\partial W_i}{\partial n}\right) a_i\right] d\Gamma(\mathbf{x}) \quad (15)$$

it is not trivial in general. This means that an eigensolution satisfies the boundary condition, $\alpha_1 u + \beta_1 t = 0$; for sure it cannot satisfy its linearly independent boundary condition, $\beta_1 u - \alpha_1 t = 0$. This theorem has been proved in Reference [13].

Let us consider another eigenproblem with the boundary condition, $\beta_1 u - \alpha_1 t = 0$, and its solution is written as

$$u = \sum_{j=1}^{\infty} W_j c_j \tag{16}$$

It can be proved that this solution will result in the rank deficiency of [**K**] and [**H**] matrices simultaneously. Now let us perform

$$[c_i]^{\mathrm{T}}[K_{ij}] = \int_{\Gamma} \left[\sum_i \left(\beta_1 W_i - \alpha_1 \frac{\partial W_i}{\partial n} \right) c_i \right] \cdot \left(\alpha_1 W_j + \beta_1 \frac{\partial W_j}{\partial n} \right) \mathrm{d}\Gamma \approx 0 \tag{17}$$

it leads to zero when the number of elements is very large. This means that the non-trivial vector [**c**] in (16) will result in the rank deficiency of the matrix [**K**] too; thus it is for sure a spurious eigensolution. It should be noticed that (17) becomes trivial because the quantity, $\sum_i (\beta_1 W_i - \alpha_1(\partial W_i/\partial n))c_i$, is zero due to the application of (16). To cancel out the spurious eigensolution, we will prove that the spurious solution, nontrivial vector [**c**] in (16), also makes the rank deficiency of the matrix [**H**]. Because (16) satisfies the boundary condition, $\beta_1 u - \alpha_1 t = 0$, we have

$$\sum_{j=1}^{\infty} \left(\beta_1 W_j - \alpha_1 \frac{\partial W_j}{\partial n} \right) c_j = 0 \tag{18}$$

It then can be concluded that

$$[c_i]^{\mathrm{T}}[H_{im}] = \int_{\Gamma_m} \left[\sum_i \left(\beta_1 W_i - \alpha_1 \frac{\partial W_i}{\partial n} \right) c_i \right] \mathrm{d}\Gamma = 0 \tag{19}$$

Therefore, the spurious eigensolution will result in the rank deficiency of the [**H**] matrix as well. Once again, (19) leads to zero because the quantity, $\sum_i (\beta_1 W_i - \alpha_1(\partial W_i/\partial n))c_i$, is zero due to the application of (16). It then can be said that the generalized singular-value decomposition method suggested in Reference [13] is available for extracting out the spurious eigensolution from [**K**] and [**H**] matrices and leaving the true eigensolution. However, such an algorithm still suffers from the ill-posed behaviour as other regular formulation does. In Reference [13], the Tikhonov's regularization method in conjunction with the generalized singular-value decomposition was introduced to treat the numerical instability. A brief introduction of such a method is given in the following.

2.3. The method to deal with numerical instability

Now let us briefly introduce the idea of treating numerical instability. We have a system as $[\mathbf{K}][\mathbf{a}] = [\mathbf{H}][\mathbf{g}]$. Since both problems can have common spurious eigensolutions, we can intuitively decomposed both matrices into the following form as

$$[\mathbf{P}][\mathbf{W}_1][\mathbf{a}] = [\mathbf{P}][\mathbf{W}_2][\mathbf{g}] = 0$$

where $[\mathbf{P}][\mathbf{W}_1] = [\mathbf{K}]$ and $[\mathbf{P}][\mathbf{W}_2] = [\mathbf{H}]$. Then the spurious eigenvalues will result in the rank deficiency of the matrix [**P**] and the true eigenvalues will result in the rank deficiency of the matrix $[\mathbf{W}_1]$ for the original problem. When the spurious eigenvalues are encountered, basically we want to extract them out by finding the matrix $[\mathbf{P}]$. That is to perform a numerical operation of L'Hospital rule on an indefinite form of 0/0. The above-mentioned technique can be achieved using the QR factorization, which is the first step of the generalized singular value decomposition.

Remember that the serious problem we encounter is not the spurious eigensolution but numerical instability of this algorithm. To treat this, we will add some small quantities into the matrices $[\mathbf{K}]$ and $[\mathbf{H}]$ to make the numerically tiny singular values occurring in both matrices become 'numerical spurious eigenvalues' such that the QR factorization can extract them out. Let $[\mathbf{K}]$ and $[\mathbf{H}]$ have the following singular value decompositions as

$$[\mathbf{K}] = [\mathbf{P}][\Sigma_1][\mathbf{V}_1^*] \tag{20a}$$

$$[\mathbf{H}] = [\mathbf{P}][\Sigma_2][\mathbf{V}_2^*] \tag{20b}$$

where $[\mathbf{P}]$ is the left unitary matrix and is the same for both matrices, $[\mathbf{V}_i^*]$ is the right unitary matrix of system i, superscript '$*$' means take the transpose and complex-conjugate of the matrix, and $[\Sigma_i]$ is a singular value matrix of system i with singular values allocated in the diagonal line. When one of the singular values is numerically very small at a specific wave number, it can be said that the system has degenerated, i.e. that the wave number is an eigenvalue. However, when a non-singular BEM is adopted there exist many numerical tiny values in the singular values, which are not true zeros. This phenomenon becomes very severe when the number of elements increases and/or a direct eigenvalue search is used at a low wave number. Now let us add two small quantities in the matrices to construct new influencing matrices as

$$[\hat{\mathbf{K}}] = [\mathbf{P}]([\Sigma_1] + \varepsilon_1[\mathbf{I}])[\mathbf{V}_1^*] \tag{21a}$$

$$[\hat{\mathbf{H}}] = [\mathbf{P}]([\Sigma_2] + \varepsilon_2[\mathbf{I}])[\mathbf{V}_2^*] \tag{21b}$$

where ε_i is the small value added to system i. The above-mentioned method is the so-called Tikhonov's regularization [19] that is commonly used to deal with an ill-posed matrix in inverse problems. The choice of the regularization parameter, ε_i, is dependent on the problem itself; however, if they are larger than the unreasonable tiny values of singular values in the original two systems, but still small enough not to overcoat the true eigenvalue, one then can successfully extract the contaminated tiny value out. If one takes the QR factorization of $[\hat{\mathbf{K}}]$ and $[\hat{\mathbf{H}}]$, the unreasonable ones can be extracted out.

2.4. Multiply connected domain and modal shape representation

It has already been mentioned in the previous section that, except for the numerical instability, the regular formulations reported so far more or less encounter two difficulties, i.e. failure in dealing with a multiply connected domain without introducing an artificial boundary and/or failure in representing mode shapes for the direct type representation. In this subsection, we focus on explaining why our proposed approach can overcome these two difficulties at the same time. First, let us look at the multiply connected domain.

Previously mentioned regular formulations resulted in trivial integral equations for a multiply connected domain. When a multiply connected domain of genus 1, as shown in Figure 1(b), is considered, the Trefftz method can easily deal with this case. One only needs to place the origin of the reference coordinate frame inside the hole. The complete T-function set involves the first kind Bessel functions and the second kind Bessel function at the same time. It is very easy to tell that for a multiply connected domain the solution should be allowed to tend to infinity value inside the hole (it is not our interested domain). This means that we have to put the second kind Bessel functions in our bases in order to represent the solution because they have infinite value at the origin. Previous methods fail simply because they do not have the second kind Bessel functions in their bases. For example, the fundamental solution used in the imaginary-part BEM in Reference [13] is simply $J_m(k\bar{r})$ in the singular integral equation (UT equation) or $J_m'(k\bar{r})$ in the hypersingular integral equation (LM equation) where $\bar{r}$ is the distance between the source point and observation point. These functions are obviously regular (finite-valued) at origin, and such a property results in failure in dealing with a multiply connected domain. On the other hand, since the Trefftz method allows use of the second kind Bessel functions it can easily overcome this difficulty.

It should be mentioned that when the proposed symmetric indirect Trefftz method is used to solve the free vibration problem of a multiply connected domain, a modification in (10) is required. For a multiply connected domain, the contaminations of higher order modes at a low wave number become severe. Let us go back to take a look of (10) now. The [**H**] matrix needs not to be a square matrix theoretically, if we choose the first i-bases in complete T-function set and have n elements on the boundary. Then the [**H**] matrix is an $i\times n$ matrix. However, in the most cases we arrange the element number and base function number to be equal such that the [**H**] matrix becomes square. This results in the difficulty for dealing with a multiply connected domain. For example, let us deal with an annular region with the Dirichlet boundary condition given on the boundaries. The true eigenequation is $J_p(kr_2)Y_p(kr_1) - J_p(kr_1)Y_p(kr_2) = 0$ where r_1 and r_2 represent for the outer radius and inner radius, respectively. When the quantity kr is very small and p tends to a large number, the calculation of multiplication of the first kind and second kind Bessel functions becomes a tragedy because at this case the value of the first kind Bessel function tends to zero. But the value of the second kind Bessel function tends to infinity. How precisely the computer can perform this calculation depends on its capacity. Remember that how a higher order mode will be encountered depends on our initial choice of the base functions. If we pick the base functions to a very high order, numerical instability cannot be overcome even when the Tikhonov's regularization method is used. Fortunately, in engineering reality we only want to know lower order modes usually and do not require higher order Bessel functions. However, we have to deal with a square matrix in computation practice. This means we have to modify (10) and this modification is introduced in the following.

Introducing the following operation:

$$[\tilde{\mathbf{H}}] \equiv [\mathbf{H}][\mathbf{H}]^{\mathrm{T}} \tag{22}$$

then both [**K**] and [$\tilde{\mathbf{H}}$] are square matrices. The symmetric property for [$\tilde{\mathbf{H}}$] matrix is obtained. This transformation means the problem we are dealing with now is

$$[\mathbf{K}][\mathbf{a}] = [\tilde{\mathbf{H}}][\tilde{\mathbf{g}}] \tag{23}$$

where $[\mathbf{g}] = [\mathbf{H}]^T[\tilde{\mathbf{g}}]$. It can be easily proved that by such an operation the spurious eigensolution existing in the original formulation now still exists in this modified formulation. After modification, we can arbitrary choose the element number as well as the base-function number. It is for sure that both the Tikhonov's regularization method and the generalized singular-value decomposition method should be used as before. Numerical examples will be given in the next section.

After discussing the multiply connected domain, we will discuss about the mode shape representation in regular formulations. In general, two kinds of regular formulations can be found. One is the direct type and the other is the indirect type. For the direct type, physical quantities on boundary are used directly. If the BEM is a regular formulation, it leads to the following expression:

$$[\mathbf{U}][\mathbf{t}] - [\mathbf{T}][\mathbf{u}] = 0 \tag{24}$$

where $[\mathbf{U}]$ and $[\mathbf{T}]$ matrices are corresponding leading coefficient matrices resulting from the direct BEM. From (24), one can obtain boundary unknown data. However, (24) does not tell us any information about inner points. The influencing matrices, $[\mathbf{U}]$ and $[\mathbf{T}]$, are built by placing the observation point and source point on the boundary. Even if we change the observation point to an inner point, (24) becomes a trivial equation. It means that the direct type regular formulation cannot construct the physical quantities inside the domain within its own formulation and thus it requires help from other formulations.

On the other hand, the indirect type BEM represents physical quantities inside the domain by a superposition of some sources. The unknowns in the resulting equation are strengths of these sources. After obtaining the strengths of sources, the solution then can be constructed easily. This merit also keeps for the regular BEM. In our opinion, the indirect type regular BEM is more practical than a direct one because it can represent modal shapes easily.

3. NUMERICAL EXAMPLES

Example 1

A circular domain with radius $R_o = 1.0$ and the Dirichlet boundary condition, $u = 0$, is given.

Fifty-one elements and 51 bases are used correspondingly. For a simply connected domain like this case, it is preferred to use $(2m+1)$ bases since we have to include $J_m(kr)\cos(m\theta)$ and $J_m(kr)\sin(m\theta)$ at the same time. By using the Tikhonov's regularization method and generalized singular-value decomposition, eigenvalues are found successfully as shown in Figure 2. In this figure, the value in the bracket is the analytical solution. The mode shapes for the first three modes are illustrated in Figures 3(a)–3(c).

Example 2

A circular domain with radius $R_o = 1.0$ and the Neumman boundary condition, $t = 0$, is given.

In this example, we can see that our method is valid for all kinds of boundary conditions. Again, fifty-one constant elements and 51 bases are used. By using the proposed method, eigenvalues are successfully found and are very close to the analytical values, as shown in Figure 4.

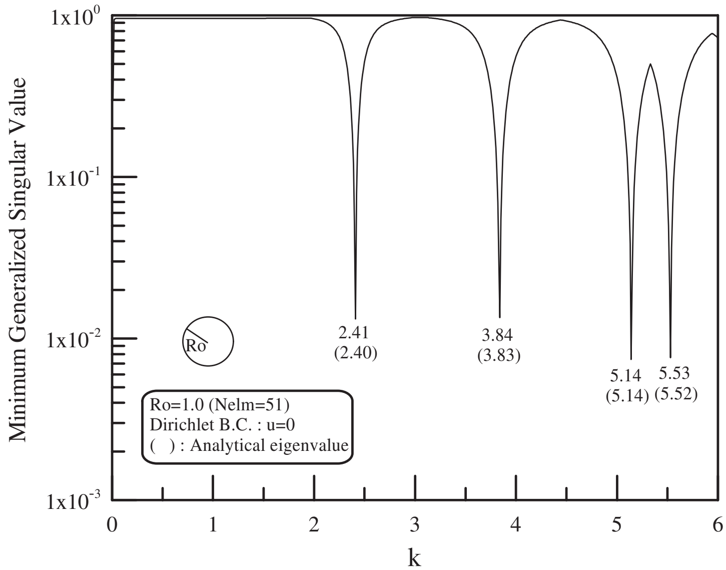


Figure 2. Eigenvalue searching for the Dirichlet boundary condition of a unit circle by using the symmetric indirect Trefftz method.

Example 3

A square membrane with edge length $L_o = 1.0$ is given, and the Neumann boundary condition, $t = 0$, is prescribed on the boundary.

In this example, a domain without radial symmetry is illustrated. Eighty-one constant elements and 81 bases are used correspondingly. It can be found in Figure 5 that the numerical results match the analytical solutions very well. At $k = 3.14$, a double root can be found by the current method as shown in Figures 6(a) and 6(b).

Example 4

An annular region with the outer radius $R_o = 1.0$ and inner radius $R_i = 0.2$ is given, and a Dirichlet boundary condition, $u = 0$, is prescribed on the boundary.

The domain is a multiply connected domain, which shows the superiority of the current approach over Kuo's method in Reference [13]. Their method is proven to fail when a multiply connected domain is treated. However, the indirect Trefftz method can easily overcome this problem by putting the origin inside the hole. In this example, we have to use the modified version as we have explained previously. The analytical values are obtained by using the eigenequation:

$$[J_m(kR_o)Y_m(kR_i) - Y_m(kR_o)J_m(kR_i)] = 0$$

We use 50 constant elements on the outer boundary and inner boundary. And we first choose to 10th order Bessel functions, both the first kind and the second kind. It means we have 42 bases totally. As shown in Figure 7, we can find that no matter how we adjust the regularization parameters in the Tikhonov's regularization method, the results are not good due to higher order mode contaminations. However, when we reduce bases from 10th order

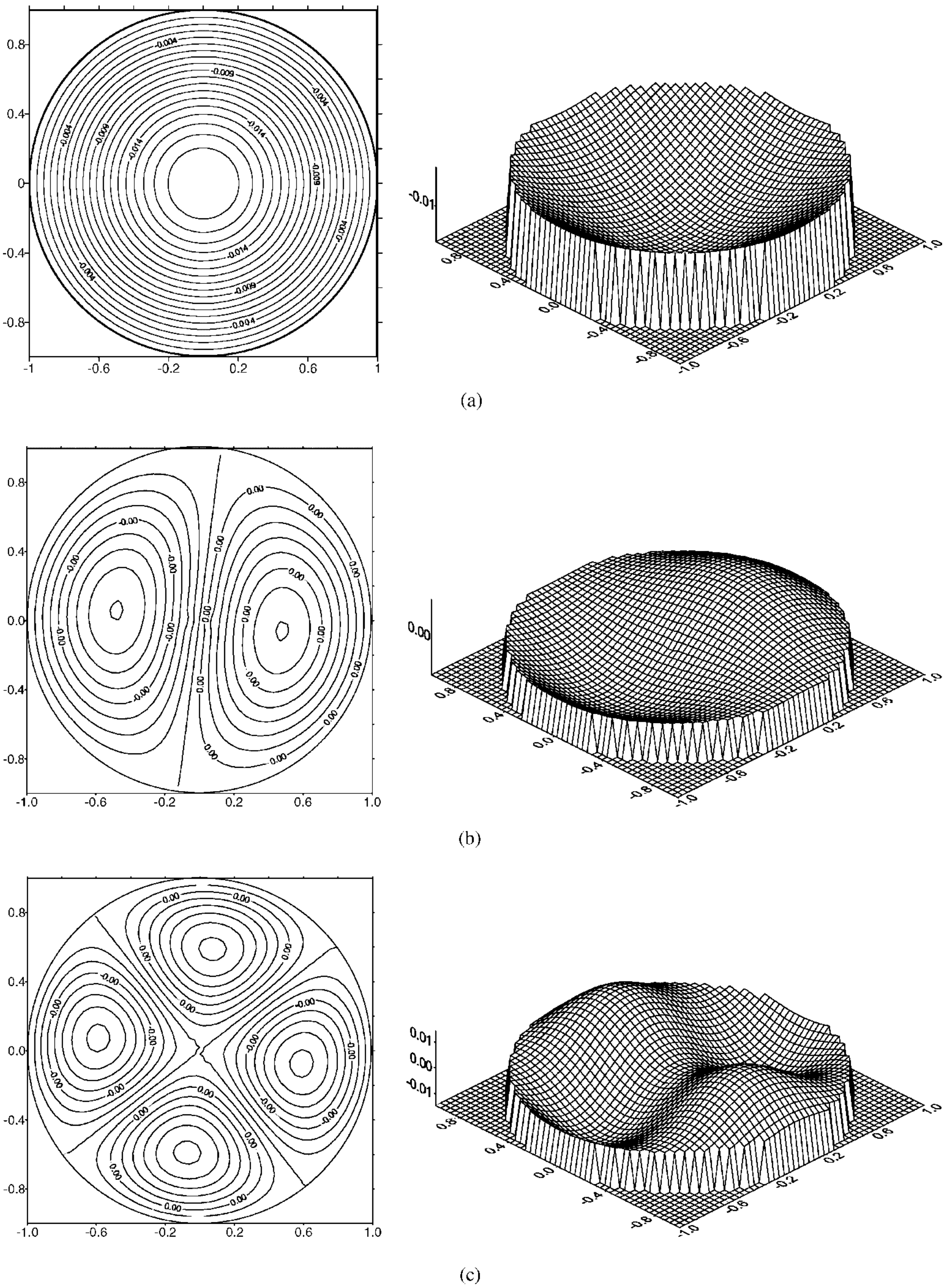


Figure 3. (a) The first mode shape of a circular membrane; (b) the second mode shape of a circular membrane; and (c) the third mode shape of a circular membrane.

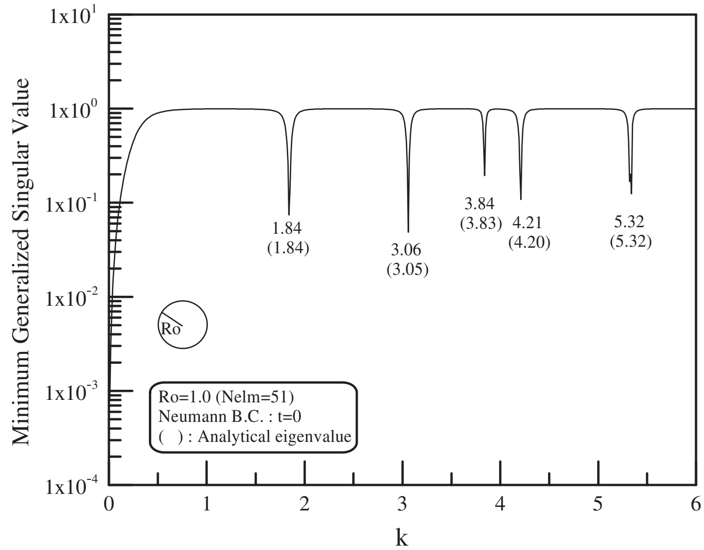


Figure 4. Eigenvalue searching for the Neumman boundary condition of a unit circle by using the symmetric indirect Trefftz method.

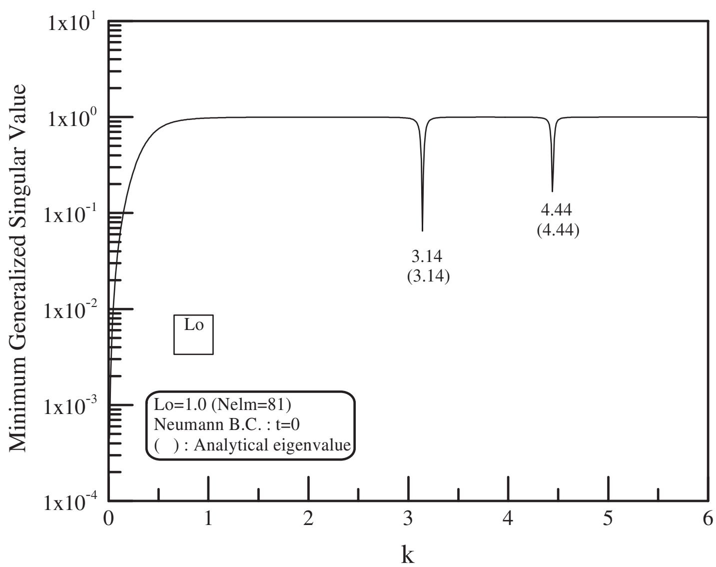
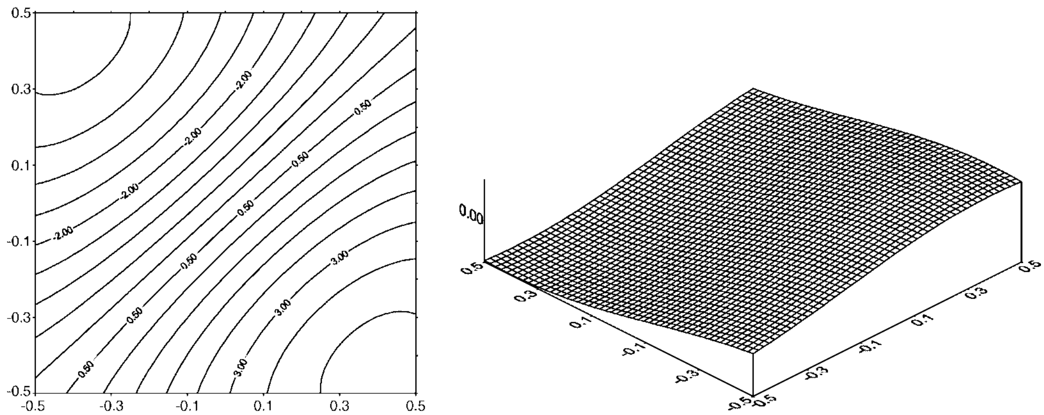
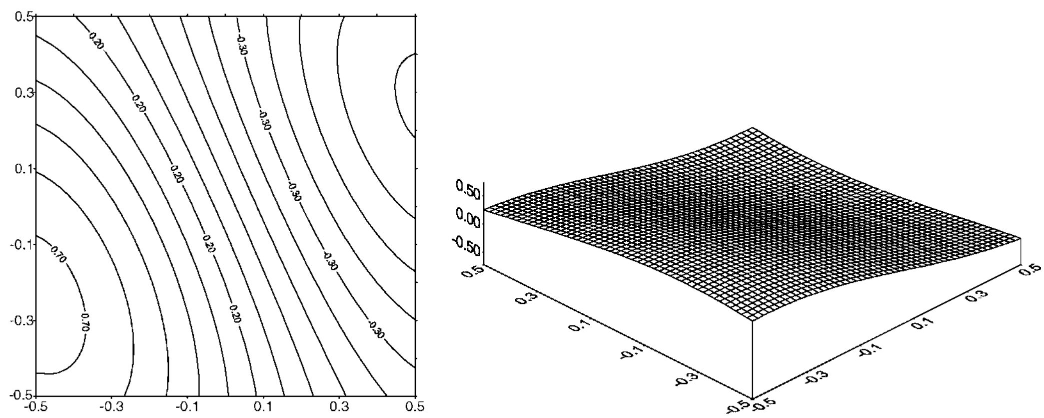


Figure 5. Eigenvalue searching for the Neumman boundary condition of a square membrane by using the symmetric indirect Trefftz method.

Bessel functions to 5th order Bessel functions (totally 22 bases are used), the solution is acceptable as shown in Figure 8. The first three mode shapes for this case are shown in Figures 9(a)–9(c).



(a)



(b)

Figure 6. A double root of a square membrane is found by using the symmetric indirect Trefftz method at wave number $k = 3.14$.

Example 5

A multiply connected domain, in which the outer boundary is a square with edge length $L_o = 2.0$ and the inner boundary is a circle with radius $R_i = 0.2$, is considered. The origin of the circular hole is the geometric centre of the whole domain. The boundary condition is the Dirichlet condition, $u = 0$.

In this example, no analytical solution is available. We compared our results with those obtained from the complex-valued dual BEM. Numerical results obtained from the indirect Trefftz method are close to those obtained from the complex-valued dual BEM as shown in Figure 10. Again we only choose the Bessel functions to the fifth order, it means that totally 22 bases are used.

Example 6

A circular domain with radius $R_o = 1.0$ is given and the Robin boundary condition, $2u + 3t = 0$, is prescribed on the boundary.

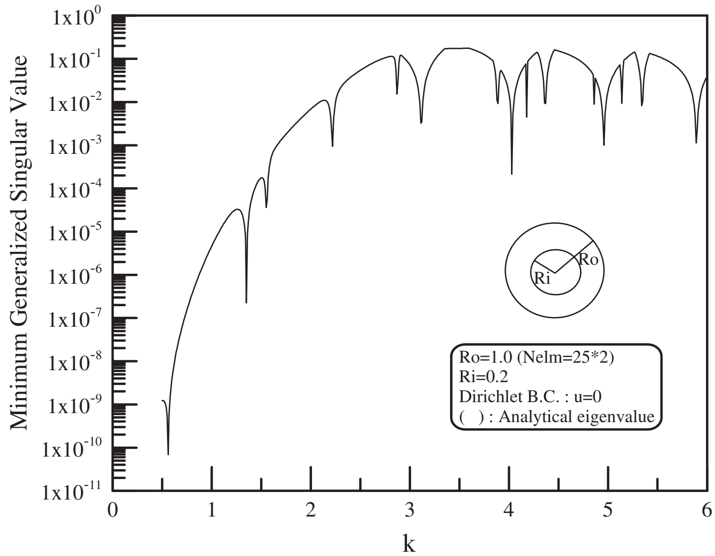


Figure 7. Some numerical contaminations still exist no matter how to adjust the Tikhonov's regularization parameters for an annular region.

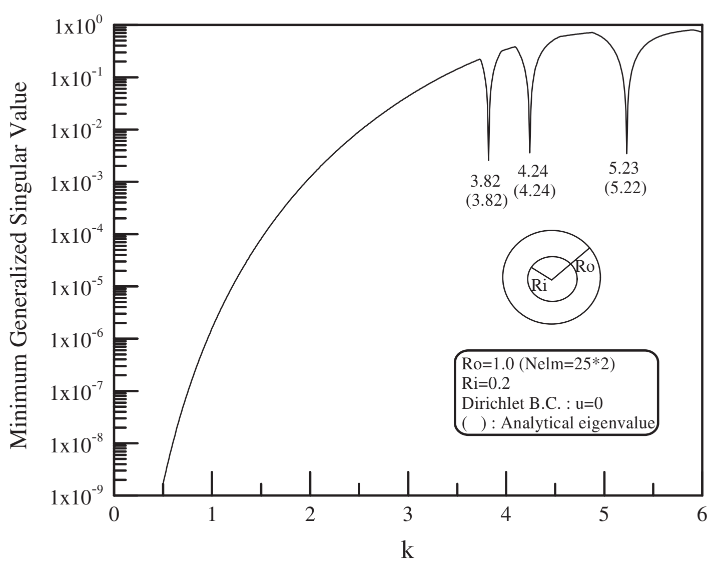
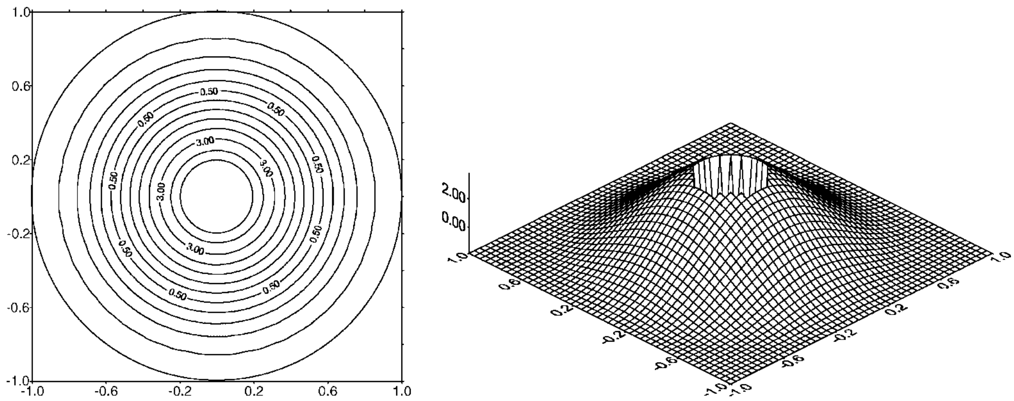
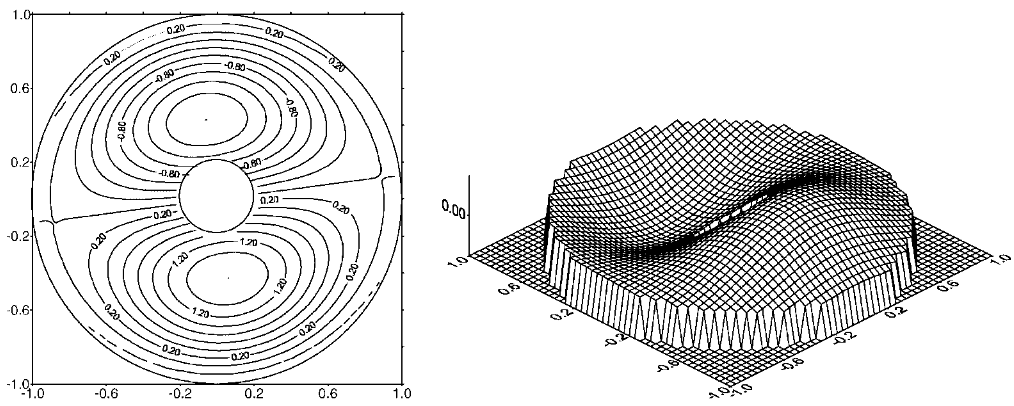


Figure 8. Numerical contaminations are eliminated after reducing the bases functions in the symmetric indirect Trefftz method.

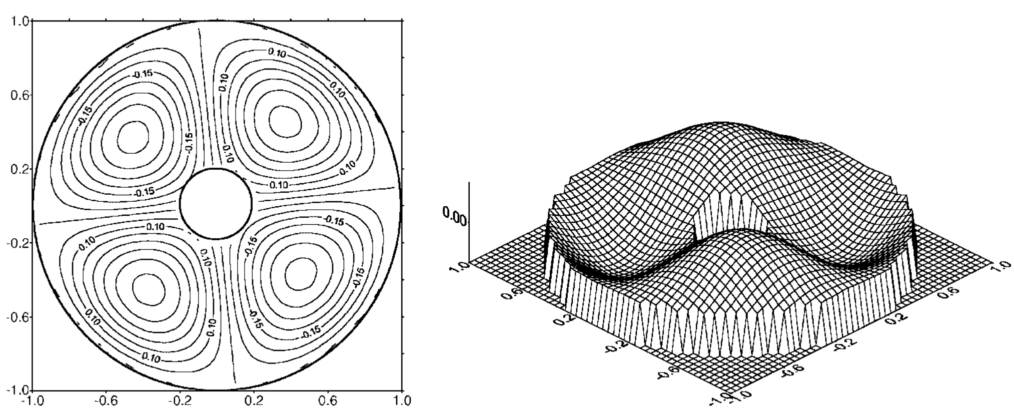
The analytical values for this case are obtained by using the true eigenequation as [13] $2J_m(kR_o) + 3J_m'(kR_o) = 0$. In this case, 51 elements and 51 bases are used correspondingly. After the Tikhonov's regularization method and generalized singular-value decomposition are adopted, eigenvalues are found successfully and numerical results match analytical solutions



(a)



(b)



(c)

Figure 9. (a) The first mode shape of an annular membrane; and (b) the second mode shape of an annular membrane; and (c) the third mode shape of an annular membrane.

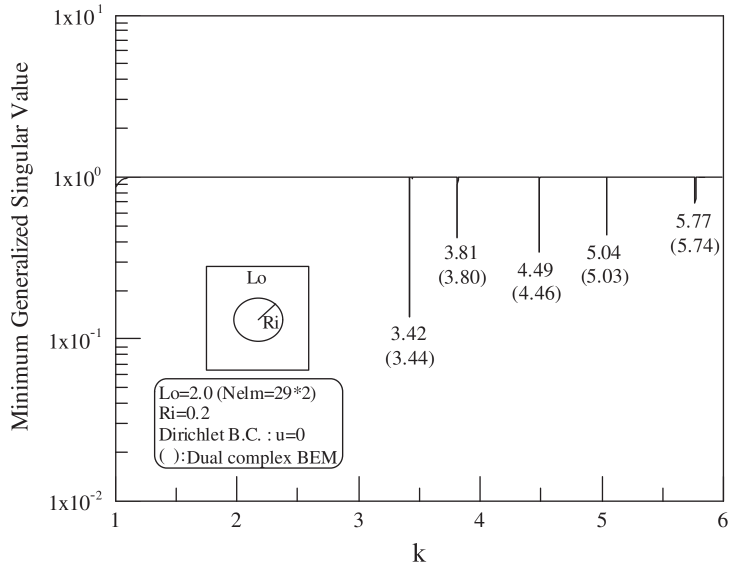


Figure 10. Eigenvalue searching for the Dirichlet boundary condition of a multiply connected domain by using the symmetric indirect Trefftz method.

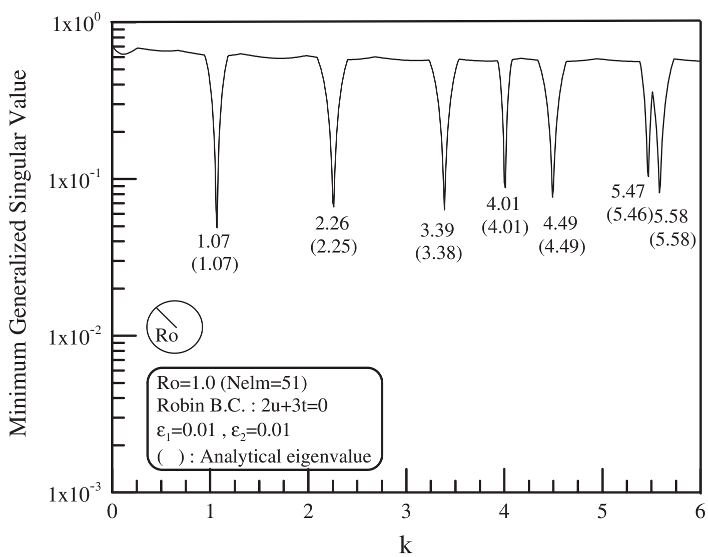


Figure 11. Eigenvalue searching for the Dirichlet boundary condition of a unit circle by using the symmetric indirect Trefftz method.

very well as shown in Figure 11. It should be noted that in this case, two regularization parameters, ε_1 and ε_2, used in the Tikhonov's regularization method for treating the ill-posed problem are both 0.01.

4. CONCLUSIONS

In this paper, a symmetric indirect Trefftz method has been developed to solve the free vibration problem of a 2D membrane. The spurious eigensolution exists in this formulation and numerical instability is encountered. To overcome these difficulties, both the generalized singular-value decomposition and Tikhonov's regularization method are used to cope with. By comparing with other regular BEM formulations, the current approach can easily treat a multiply connected domain of genus 1 and it can represent the modal shape within its own formulation. Numerical examples have shown the validity of the proposed approach.

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