INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING
Int. J. Numer. Meth. Engng 2006; **65**:236–263
Published online 6 October 2005 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/nme.1498

Mathematical analysis and numerical study to free vibrations of annular plates using BIEM and BEM

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SUMMARY

In this paper, the spurious eigenequations for annular plate eigenproblems by using BIEM and BEM are studied in the continuous and discrete systems. Since any two boundary integral equations in the plate formulation (4 equations) can be chosen, 6 (C_2^4) options can be considered instead of only two approaches (single-layer and double-layer methods, or singular and hypersingular equations) which are adopted for the eigenproblems of the membrane and acoustic problems. The occurring mechanism of the spurious eigenequation for annular plates in the complex-valued formulations is studied analytically. For the continuous system, degenerate kernels for the fundamental solution and the Fourier series expansion for the circular boundary density are employed to derive the true and spurious eigenequations analytically. For the discrete system, the degenerate kernels for the fundamental solution and circulants resulting from the circular boundary are employed to determine the true and spurious eigenequations. True eigenequation depends on the specified boundary condition while spurious eigenequation is embedded in each formulation. It is found that the spurious eigenvalue for the annular plate is the true eigenvalue of the associated interior problem with an inner radius of the annular domain. Also, we provide three methods (SVD updating technique, Burton and Miller method and CHIEF method) to suppress the occurrence of the spurious eigenvalues. Several examples were demonstrated to check the validity of the formulations. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: boundary integral equation method; boundary element method; annular plate; spurious eigenvalue; degenerate kernel; Fourier series; circulants; SVD technique of updating term; Burton and Miller method; CHIEF method

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Contract/grant sponsor: National Science Council; contract/grant number: NSC-92-2211-E-019-013

Received 20 September 2004
Revised 27 April 2005
Accepted 12 July 2005

1. INTRODUCTION

For the eigenproblems, either the real-part or imaginary-part BEM instead of complex-valued BEM results in spurious eigenequations. Tai and Shaw [1] first employed BEM to solve membrane vibration using a complex-valued kernel. De Mey [2, 3] and Hutchinson and Wong [4, 5] employed only the real-part kernel to solve the membrane and plate vibrations, free of the complex-valued computation in sacrifice of occurrence of spurious eigenequations. Wong and Hutchinson [4] have used a direct BEM for solving plate vibration involving displacement, slope, moment and shear force. They were able to obtain eigenvalues for the clamped plates by employing only the real-part BEM with obvious computational gains. However, this saving leads to the spurious eigenvalues in addition to the true ones for free vibration analysis. This is the reason why Chen and his coworkers have developed many systematic techniques, e.g. dual formulation, domain partition, SVD updating technique [6], CHEEF method [7], for sorting out the true and the spurious eigenvalues. Niwa *et al.* [8] also stated that 'One must take care to use the complete Green's function for outgoing waves, as attempts to use just the real (singular) or imaginary (regular) part separately will not provide the complete spectrum'. As quoted from the reply of Hutchinson [9], this comment is not correct since the real-part or imaginary-part BEM does not lose any true eigenvalue. The reason is that the real-part and imaginary-part kernels satisfy the Hilbert transform pair. They are not fully independent. To use both parts, real and imaginary kernels may not be economical. Complete eigenspectrum is imbedded in either one, real or imaginary-part kernel. The Hilbert transform is the constraint in the frequency domain corresponding to the causal effect in the time-domain fundamental solution. Tai and Shaw [1] claimed that spurious eigenvalues are not present if the complex-valued kernel is employed for the eigenproblem. However, it is true only for the case of problem with a simply connected domain. For multiply connected problems, spurious eigenvalues still appear even though the complex-valued BEM is utilized. This finding and the treatment for spurious eigenvalues have been verified in the membrane and acoustic problems [10, 11]. The spurious eigenvalues occurs in two aspects: one is for the simply connected eigenproblem by using the real-part or imaginary-part BEM; the other is for the multiply connected eigenproblem even though the complex-valued BEM is utilized.

In this paper, the eigenproblem for the annular plate is solved by using the boundary integral equation method (BIEM) as well as the boundary element method (BEM). The true and spurious eigenequations are derived by using the complex-valued BEM. The occurring mechanism of the spurious eigenequation for the plate eigenproblem in each formulation is studied analytically in both the continuous and discrete models. Three alternatives, SVD updating technique, Burton and Miller method and CHIEF method are utilized to suppress the occurrence of the spurious eigenvalues. Plates subject to three types of boundary conditions, clamped, simply supported and free boundary conditions, are demonstrated. Analytical derivations and numerical results are illustrated to check the validity of the present formulations.

2. BOUNDARY INTEGRAL FORMULATION AND BOUNDARY ELEMENT METHOD FOR PLATE EIGENPROBLEMS

The governing equation for the free flexural vibration of a uniform thin plate is written as follows:

$$\nabla^4 u(x) = \lambda^4 u(x), \quad x \in \Omega \tag{1}$$

where u is the lateral displacement, $\lambda^4 = \omega^2 \rho_0 h/D$, λ is the frequency parameter, ω is the circular frequency, ρ_0 is the surface density, D is the flexural rigidity expressed as $D = Eh^3/[12(1-\nu^2)]$ in terms of Young's modulus E, the Poisson ratio ν and the plate thickness h, and Ω is the domain. The integral equations for the domain point can be derived from the Rayleigh–Green identity [12] as follows:

$$\begin{aligned} u(x) = & -\int_B U(s,x)v(s)\,\mathrm{d}B(s) + \int_B \Theta(s,x)m(s)\,\mathrm{d}B(s) \\ & -\int_B M(s,x)\theta(s)\,\mathrm{d}B(s) + \int_B V(s,x)u(s)\,\mathrm{d}B(s), \quad x \in \Omega \end{aligned} \tag{2}$$

$$\begin{aligned} \theta(x) = & -\int_B U_\theta(s,x)v(s)\,\mathrm{d}B(s) + \int_B \Theta_\theta(s,x)m(s)\,\mathrm{d}B(s) \\ & -\int_B M_\theta(s,x)\theta(s)\,\mathrm{d}B(s) + \int_B V_\theta(s,x)u(s)\,\mathrm{d}B(s), \quad x \in \Omega \end{aligned} \tag{3}$$

$$\begin{aligned} m(x) = & -\int_B U_m(s,x)v(s)\,\mathrm{d}B(s) + \int_B \Theta_m(s,x)m(s)\,\mathrm{d}B(s) \\ & -\int_B M_m(s,x)\theta(s)\,\mathrm{d}B(s) + \int_B V_m(s,x)u(s)\,\mathrm{d}B(s), \quad x \in \Omega \end{aligned} \tag{4}$$

$$\begin{aligned} v(x) = & -\int_B U_v(s,x)v(s)\,\mathrm{d}B(s) + \int_B \Theta_v(s,x)m(s)\,\mathrm{d}B(s) \\ & -\int_B M_v(s,x)\theta(s)\,\mathrm{d}B(s) + \int_B V_v(s,x)u(s)\,\mathrm{d}B(s), \quad x \in \Omega \end{aligned} \tag{5}$$

where B is the boundary, u, θ, m and v mean the displacement, slope, normal moment, effective shear force, s and x are the source and field points, respectively, U, Θ, M and V kernel functions will be elaborated on later. The kernel function $U(s,x)$ is the fundamental solution which satisfies

$$\nabla^4 U(s,x) - \lambda^4 U(s,x) = \delta(x-s) \tag{6}$$

where $\delta(x-s)$ is the Dirac-delta function. Considering the two singular solutions ($Y_0(\lambda r)$ and $K_0(\lambda r)$ [13], which are the zeroth-order of the second-kind Bessel and modified Bessel functions, respectively) and two regular solutions ($J_0(\lambda r)$ and $I_0(\lambda r)$, which are the zeroth-order of the first-kind Bessel and modified Bessel functions, respectively) in the fundamental solution, we have

$$U(s,x) = \frac{1}{8\lambda^2}\left[(Y_0(\lambda r) + \mathrm{i}J_0(\lambda r)) + \frac{2}{\pi}(K_0(\lambda r) + \mathrm{i}I_0(\lambda r))\right] \tag{7}$$

where $r \equiv |s-x|$ and $\mathrm{i}^2 = -1$. The other three kernels, $\Theta(s,x)$, $M(s,x)$ and $V(s,x)$, are defined as follows:

$$\Theta(s,x) = \mathscr{K}_\theta(U(s,x)) \tag{8}$$

$$M(s,x) = \mathscr{K}_m(U(s,x)) \tag{9}$$

$$V(s,x) = \mathscr{K}_v(U(s,x)) \tag{10}$$

where $\mathscr{K}_\theta(\cdot)$, $\mathscr{K}_m(\cdot)$ and $\mathscr{K}_v(\cdot)$ mean the operators defined by

$$\mathscr{K}_\theta(\cdot) \equiv \frac{\partial(\cdot)}{\partial n} \tag{11}$$

$$\mathscr{K}_m(\cdot) \equiv v\nabla^2(\cdot) + (1-v)\frac{\partial^2(\cdot)}{\partial n^2} \tag{12}$$

$$\mathscr{K}_v(\cdot) \equiv \frac{\partial \nabla^2(\cdot)}{\partial n} + (1-v)\frac{\partial}{\partial t}\left[\left(\frac{\partial^2(\cdot)}{\partial n \partial t}\right)\right] \tag{13}$$

where $\frac{\partial}{\partial n}$ and $\frac{\partial}{\partial t}$ are the normal and tangential derivatives, respectively. The displacement, slope, normal moment and effective shear force are derived by

$$\theta(x) = \mathscr{K}_\theta(u(x)) \tag{14}$$

$$m(x) = \mathscr{K}_m(u(x)) \tag{15}$$

$$v(x) = \mathscr{K}_v(u(x)) \tag{16}$$

Once the field point x locates outside the interested domain, the null-field BIEs of the direct method in Equations (2)–(5) yield

$$\begin{aligned} 0 = &-\int_B U(s,x)v(s)\,\mathrm{d}B(s) + \int_B \Theta(s,x)m(s)\,\mathrm{d}B(s) \\ &-\int_B M(s,x)\theta(s)\,\mathrm{d}B(s) + \int_B V(s,x)u(s)\,\mathrm{d}B(s), \quad x \in \Omega^e \end{aligned} \tag{17}$$

$$\begin{aligned} 0 = &-\int_B U_\theta(s,x)v(s)\,\mathrm{d}B(s) + \int_B \Theta_\theta(s,x)m(s)\,\mathrm{d}B(s) \\ &-\int_B M_\theta(s,x)\theta(s)\,\mathrm{d}B(s) + \int_B V_\theta(s,x)u(s)\,\mathrm{d}B(s), \quad x \in \Omega^e \end{aligned} \tag{18}$$

$$0=-\int_B U_m(s,x)v(s)\,\mathrm{d}B(s)+\int_B \Theta_m(s,x)m(s)\,\mathrm{d}B(s)$$
$$-\int_B M_m(s,x)\theta(s)\,\mathrm{d}B(s)+\int_B V_m(s,x)u(s)\,\mathrm{d}B(s),\quad x\in\Omega^e \tag{19}$$

$$0=-\int_B U_v(s,x)v(s)\,\mathrm{d}B(s)+\int_B \Theta_v(s,x)m(s)\,\mathrm{d}B(s)$$
$$-\int_B M_v(s,x)\theta(s)\,\mathrm{d}B(s)+\int_B V_v(s,x)u(s)\,\mathrm{d}B(s),\quad x\in\Omega^e \tag{20}$$

where Ω^e is the complementary domain. Although the null-field BIEs are not singular due to $x\neq s$, they are indeed used for the point x near the boundary by using the appropriate forms of degenerate kernels in real computations. Improper integrals can be avoided by using the appropriate expressions of degenerate kernels.

3. ANALYTICAL DERIVATION OF THE TRUE AND SPURIOUS EIGENEQUATIONS OF AN ANNULAR PLATE IN BIEM (CONTINUOUS SYSTEM) AND BEM (DISCRETE SYSTEM)

3.1. Continuous system for BIEM

Case 1: Annular plate clamped on both the outer and inner boundaries (C–C).
We consider an annular plate clamped on the outer circle B_1 ($u_1=0$ and $\theta_1=0$) and the inner circle B_2 ($u_2=0$ and $\theta_2=0$), where u_1,θ_1,u_2 and θ_2 are the displacement and slope on the B_1 and B_2, respectively. The radii of the outer and inner circles are a and b, respectively. The moment and shear force, $m_1(s), m_2(s), v_1(s)$ and $v_2(s)$, along the circular boundary can be expanded into the Fourier series by

$$m_1(s)=\sum_{n=0}^{\infty}(p_{1,n}^{\mathrm{cc}}\cos(n\bar{\phi})+q_{1,n}^{\mathrm{cc}}\sin(n\bar{\phi})),\quad s\in B_1 \tag{21}$$

$$m_2(s)=\sum_{n=0}^{\infty}(p_{2,n}^{\mathrm{cc}}\cos(n\bar{\phi})+q_{2,n}^{\mathrm{cc}}\sin(n\bar{\phi})),\quad s\in B_2 \tag{22}$$

$$v_1(s)=\sum_{n=0}^{\infty}(a_{1,n}^{\mathrm{cc}}\cos(n\bar{\phi})+b_{1,n}^{\mathrm{cc}}\sin(n\bar{\phi})),\quad s\in B_1 \tag{23}$$

$$v_2(s)=\sum_{n=0}^{\infty}(a_{2,n}^{\mathrm{cc}}\cos(n\bar{\phi})+b_{2,n}^{\mathrm{cc}}\sin(n\bar{\phi})),\quad s\in B_2 \tag{24}$$

where the superscript 'cc' denotes the clamped–clamped case, $\bar{\phi}$ is the angle on the circular boundary, $a_{i,n}^{\mathrm{cc}}$, $b_{i,n}^{\mathrm{cc}}$, $p_{i,n}^{\mathrm{cc}}$ and $q_{i,n}^{\mathrm{cc}}$ $(i=1,2)$ are the unknown Fourier coefficients on B_i $(i=1,2)$. When the null-field point locates near B_1^+, substitution of Equations (21)–(24)

into Equations (17) and (18) yields

$$
\begin{aligned}
0 = &-\int_{B_1} U(s,x)\left[\sum_{n=0}^{\infty}(a_{1,n}^{cc}\cos(n\bar{\phi})+b_{1,n}^{cc}\sin(n\bar{\phi}))\right]\mathrm{d}B(s)\\
&-\int_{B_2} U(s,x)\left[\sum_{n=0}^{\infty}(a_{2,n}^{cc}\cos(n\bar{\phi})+b_{2,n}^{cc}\sin(n\bar{\phi}))\right]\mathrm{d}B(s)\\
&+\int_{B_1} \Theta(s,x)\left[\sum_{n=0}^{\infty}(p_{1,n}^{cc}\cos(n\bar{\phi})+q_{1,n}^{cc}\sin(n\bar{\phi}))\right]\mathrm{d}B(s)\\
&+\int_{B_2} \Theta(s,x)\left[\sum_{n=0}^{\infty}(p_{2,n}^{cc}\cos(n\bar{\phi})+q_{2,n}^{cc}\sin(n\bar{\phi}))\right]\mathrm{d}B(s), \quad x\rightarrow B_1^+
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
0 = &-\int_{B_1} U_\theta(s,x)\left[\sum_{n=0}^{\infty}(a_{1,n}^{cc}\cos(n\bar{\phi})+b_{1,n}^{cc}\sin(n\bar{\phi}))\right]\mathrm{d}B(s)\\
&-\int_{B_2} U_\theta(s,x)\left[\sum_{n=0}^{\infty}(a_{2,n}^{cc}\cos(n\bar{\phi})+b_{2,n}^{cc}\sin(n\bar{\phi}))\right]\mathrm{d}B(s)\\
&+\int_{B_1} \Theta_\theta(s,x)\left[\sum_{n=0}^{\infty}(p_{1,n}^{cc}\cos(n\bar{\phi})+q_{1,n}^{cc}\sin(n\bar{\phi}))\right]\mathrm{d}B(s)\\
&+\int_{B_2} \Theta_\theta(s,x)\left[\sum_{n=0}^{\infty}(p_{2,n}^{cc}\cos(n\bar{\phi})+q_{2,n}^{cc}\sin(n\bar{\phi}))\right]\mathrm{d}B(s), \quad x\rightarrow B_1^+
\end{aligned} \tag{26}
$$

The kernels in Equations (25) and (26) must be carefully chosen using the interior and exterior expressions of degenerate kernels. This is the key to avoid the improper integrals. When the null-field point locates near B_2^-, substitution of Equations (21)–(24) into Equations (17) and (18) yields

$$
\begin{aligned}
0 = &-\int_{B_1} U(s,x)\left[\sum_{n=0}^{\infty}(a_{1,n}^{cc}\cos(n\bar{\phi})+b_{1,n}^{cc}\sin(n\bar{\phi}))\right]\mathrm{d}B(s)\\
&-\int_{B_2} U(s,x)\left[\sum_{n=0}^{\infty}(a_{2,n}^{cc}\cos(n\bar{\phi})+b_{2,n}^{cc}\sin(n\bar{\phi}))\right]\mathrm{d}B(s)\\
&+\int_{B_1} \Theta(s,x)\left[\sum_{n=0}^{\infty}(p_{1,n}^{cc}\cos(n\bar{\phi})+q_{1,n}^{cc}\sin(n\bar{\phi}))\right]\mathrm{d}B(s)\\
&+\int_{B_2} \Theta(s,x)\left[\sum_{n=0}^{\infty}(p_{2,n}^{cc}\cos(n\bar{\phi})+q_{2,n}^{cc}\sin(n\bar{\phi}))\right]\mathrm{d}B(s), \quad x\rightarrow B_2^-
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
0 = &-\int_{B_1} U_\theta(s,x)\left[\sum_{n=0}^{\infty}(a_{1,n}^{cc}\cos(n\bar{\phi})+b_{1,n}^{cc}\sin(n\bar{\phi}))\right]\mathrm{d}B(s)\\
&-\int_{B_2} U_\theta(s,x)\left[\sum_{n=0}^{\infty}(a_{2,n}^{cc}\cos(n\bar{\phi})+b_{2,n}^{cc}\sin(n\bar{\phi}))\right]\mathrm{d}B(s)
\end{aligned}
$$

$$+\int_{B_1} \Theta_\theta(s,x)\left[\sum_{n=0}^{\infty}(p_{1,n}^{cc}\cos(n\bar{\phi})+q_{1,n}^{cc}\sin(n\bar{\phi}))\right]\mathrm{d}B(s)$$

$$+\int_{B_2} \Theta_\theta(s,x)\left[\sum_{n=0}^{\infty}(p_{2,n}^{cc}\cos(n\bar{\phi})+q_{2,n}^{cc}\sin(n\bar{\phi}))\right]\mathrm{d}B(s), \quad x\to B_2^- \tag{28}$$

Similarly, Equations (27) and (28) are free of singular integrals by choosing the appropriated kernels of degenerate kernels. The kernel functions, $U(s,x)$, $\Theta(s,x)$, $U_\theta(s,x)$ and $\Theta_\theta(s,x)$, can be expressed by using the expansion formulae,

$$Y_0(\lambda r)=\begin{cases} Y_0^{\mathrm{i}}(\lambda r)=\sum\limits_{m=-\infty}^{\infty} Y_m(\lambda\bar{\rho})J_m(\lambda\rho)\cos(m(\bar{\phi}-\phi)), & \bar{\rho}\geqslant\rho \\ Y_0^{\mathrm{e}}(\lambda r)=\sum\limits_{m=-\infty}^{\infty} Y_m(\lambda\rho)J_m(\lambda\bar{\rho})\cos(m(\bar{\phi}-\phi)), & \rho>\bar{\rho}\end{cases} \tag{29}$$

$$K_0(\lambda r)=\begin{cases} K_0^{\mathrm{i}}(\lambda r)=\sum\limits_{m=-\infty}^{\infty} K_m(\lambda\bar{\rho})I_m(\lambda\rho)\cos(m(\bar{\phi}-\phi)), & \bar{\rho}\geqslant\rho \\ K_0^{\mathrm{e}}(\lambda r)=\sum\limits_{m=-\infty}^{\infty} K_m(\lambda\rho)I_m(\lambda\bar{\rho})\cos(m(\bar{\phi}-\phi)), & \rho>\bar{\rho}\end{cases} \tag{30}$$

$$J_0(\lambda r)=\begin{cases} J_0^{\mathrm{i}}(\lambda r)=\sum\limits_{m=-\infty}^{\infty} J_m(\lambda\bar{\rho})J_m(\lambda\rho)\cos(m(\bar{\phi}-\phi)), & \bar{\rho}\geqslant\rho \\ J_0^{\mathrm{e}}(\lambda r)=\sum\limits_{m=-\infty}^{\infty} J_m(\lambda\rho)J_m(\lambda\bar{\rho})\cos(m(\bar{\phi}-\phi)), & \rho>\bar{\rho}\end{cases} \tag{31}$$

$$I_0(\lambda r)=\begin{cases} I_0^{\mathrm{i}}(\lambda r)=\sum\limits_{m=-\infty}^{\infty} (-1)^m I_m(\lambda\bar{\rho})I_m(\lambda\rho)\cos(m(\bar{\phi}-\phi)), & \bar{\rho}\geqslant\rho \\ I_0^{\mathrm{e}}(\lambda r)=\sum\limits_{m=-\infty}^{\infty} (-1)^m I_m(\lambda\rho)I_m(\lambda\bar{\rho})\cos(m(\bar{\phi}-\phi)), & \rho>\bar{\rho}\end{cases} \tag{32}$$

where J_m and I_m denote the mth-order Bessel and modified Bessel functions of the first kind, Y_m and K_m denote the mth-order Bessel and modified Bessel functions of the second kind. The superscripts 'i' and 'e' denote the interior point $(\bar{\rho}>\rho)$ and the exterior point $(\bar{\rho}<\rho)$, $s=(\bar{\rho},\bar{\phi})$ and $x=(\rho,\phi)$ are the polar coordinates of s and x, respectively. Similarly, the other kernels can be expanded into degenerate forms. By using the degenerate kernels into Equations (25)–(28) and by employing the orthogonality condition of the Fourier series, the Fourier coefficients $a_{i,n}^{cc}$ and $p_{i,n}^{cc}$ $(i=1,2)$ satisfy

$$[TM_n^{cc}]_{4\times 4}\begin{Bmatrix} a_{1,n}^{cc} \\ a_{2,n}^{cc} \\ p_{1,n}^{cc} \\ p_{2,n}^{cc}\end{Bmatrix}_{4\times 1} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0\end{Bmatrix}_{4\times 1} \tag{33}$$

For the existence of non-trivial solution for the generalized coefficients of $a_{i,n}^{\mathrm{cc}}$, $p_{i,n}^{\mathrm{cc}}$, $b_{i,n}^{\mathrm{cc}}$ and $q_{i,n}^{\mathrm{cc}}$ $(i=1,2)$, the matrix has a zero determinant, i.e.

$$\det[TM_n^{\mathrm{cc}}]=0 \tag{34}$$

By using the properties of the determinant, we have

$$\det[TM_n^{\mathrm{cc}}]=C_1\det([S_n^{u\theta}][T_n^{\mathrm{cc}}]),\quad C_1 \text{ is a constant} \tag{35}$$

where

$$[S_n^{u\theta}]_{4\times 4}=\begin{bmatrix}(Y_n(\lambda a)+\mathrm{i}J_n(\lambda a)) & 0 & (K_n(\lambda a)+\mathrm{i}I_n(\lambda a)) & 0\\ \mathrm{i}J_n(\lambda b) & J_n(\lambda b) & \mathrm{i}I_n(\lambda b) & I_n(\lambda b)\\ (Y_n'(\lambda a)+\mathrm{i}J_n'(\lambda a)) & 0 & (K_n'(\lambda a)+\mathrm{i}I_n'(\lambda a)) & 0\\ \mathrm{i}J_n'(\lambda b) & J_n'(\lambda b) & \mathrm{i}I_n'(\lambda b) & I_n'(\lambda b)\end{bmatrix}_{4\times 4} \tag{36}$$

and

$$[T_n^{\mathrm{cc}}]_{4\times 4}=\begin{bmatrix}J_n(\lambda a) & J_n(\lambda b) & J_n'(\lambda a) & J_n'(\lambda b)\\ Y_n(\lambda a) & Y_n(\lambda b) & Y_n'(\lambda a) & Y_n'(\lambda b)\\ I_n(\lambda a) & I_n(\lambda b) & I_n'(\lambda a) & I_n'(\lambda b)\\ K_n(\lambda a) & K_n(\lambda b) & K_n'(\lambda a) & K_n'(\lambda b)\end{bmatrix}_{4\times 4} \tag{37}$$

It is noted that the matrix $[T_n^{\mathrm{cc}}]$ denotes the matrix of true eigenequation for the C–C case and the matrix $[S_n^{u\theta}]$ denotes the matrix of spurious eigenequation in the u,θ formulation. Zero determinant in Equation (35) implies that the eigenequation is

$$\det([S_n^{u\theta}][T_n^{\mathrm{cc}}])=0,\quad n=0,\pm 1,\pm 2,\ldots,\pm(N-1),N \tag{38}$$

After comparing with the analytical solution for the annular plate [14], the former matrix $[S_n^{u\theta}]$ in Equation (38) results in the spurious eigenequation while the latter matrix $[T_n^{\mathrm{cc}}]$ results in the true eigenequation. The spurious eigenequation in Equation (36) will be elaborated on later.

Case 2: Annular plate simply supported on both the outer and inner boundaries.
Following the same procedure of case 1, we have

$$\det[TM_n^{\mathrm{ss}}]=C_2\det([S_n^{u\theta}][T_n^{\mathrm{ss}}]),\quad C_2 \text{ is a constant} \tag{39}$$

where

$$[T_n^{\mathrm{ss}}]_{4\times 4}=\begin{bmatrix}J_n(\lambda a) & J_n(\lambda b) & \alpha_n^J(\lambda a) & \alpha_n^J(\lambda b)\\ Y_n(\lambda a) & Y_n(\lambda b) & \alpha_n^Y(\lambda a) & \alpha_n^Y(\lambda b)\\ I_n(\lambda a) & I_n(\lambda b) & \alpha_n^I(\lambda a) & \alpha_n^I(\lambda b)\\ K_n(\lambda a) & K_n(\lambda b) & \alpha_n^K(\lambda a) & \alpha_n^K(\lambda b)\end{bmatrix}_{4\times 4} \tag{40}$$

in which $\alpha_n^J(\cdot)$, $\alpha_n^Y(\cdot)$, $\alpha_n^I(\cdot)$ and $\alpha_n^K(\cdot)$ are listed in Appendix A. It is noted that the matrix $[T_n^{\mathrm{ss}}]$ denotes the matrix of true eigenequation for the S–S case. Zero determinant in Equation (39) implies that the eigenequation is

$$\det([S_n^{u\theta}][T_n^{\mathrm{ss}}]) = 0, \quad n = 0, \pm 1, \pm 2, \ldots, \pm(N-1), N \tag{41}$$

After comparing with the analytical solution for the annular plate [14], the former matrix $[S_n^{u\theta}]$ in Equation (41) is the same as Equation (36) which results in the spurious eigenequation while the latter matrix $[T_n^{\mathrm{ss}}]$ results in the true eigenequation.

Case 3: Annular plate free on both the outer and inner boundaries.
Similarly, we have

$$\det[TM_n^{\mathrm{ff}}] = C_3 \det([S_n^{u\theta}][T_n^{\mathrm{ff}}]), \quad C_3 \text{ is a constant} \tag{42}$$

where

$$[T_n^{\mathrm{ff}}] = \begin{bmatrix} \alpha_n^J(\lambda a) & \alpha_n^J(\lambda b) & \beta_n^J(\lambda a) + \dfrac{(1-\nu)}{a}\gamma_n^J(\lambda a) & \beta_n^J(\lambda b) + \dfrac{(1-\nu)}{b}\gamma_n^J(\lambda b) \\ \alpha_n^Y(\lambda a) & \alpha_n^Y(\lambda b) & \beta_n^Y(\lambda a) + \dfrac{(1-\nu)}{a}\gamma_n^Y(\lambda a) & \beta_n^Y(\lambda b) + \dfrac{(1-\nu)}{b}\gamma_n^Y(\lambda b) \\ \alpha_n^I(\lambda a) & \alpha_n^I(\lambda b) & \beta_n^I(\lambda a) + \dfrac{(1-\nu)}{a}\gamma_n^I(\lambda a) & \beta_n^I(\lambda b) + \dfrac{(1-\nu)}{b}\gamma_n^I(\lambda b) \\ \alpha_n^K(\lambda a) & \alpha_n^K(\lambda b) & \beta_n^K(\lambda a) + \dfrac{(1-\nu)}{a}\gamma_n^K(\lambda a) & \beta_n^K(\lambda b) + \dfrac{(1-\nu)}{b}\gamma_n^K(\lambda b) \end{bmatrix} \tag{43}$$

in which $\beta_n^J(\cdot)$, $\beta_n^Y(\cdot)$, $\beta_n^I(\cdot)$, $\beta_n^K(\cdot)$, $\gamma_n^J(\cdot)$, $\gamma_n^Y(\cdot)$, $\gamma_n^I(\cdot)$ and $\gamma_n^K(\cdot)$ are listed in Appendix A. It is noted that the matrix $[T_n^{\mathrm{ff}}]$ denotes the matrix of true eigenequation for the F–F case. Zero determinant in Equation (42) implies that the eigenequation is

$$\det([S_n^{u\theta}][T_n^{\mathrm{ff}}]) = 0, \quad n = 0, \pm 1, \pm 2, \ldots, \pm(N-1), N \tag{44}$$

After comparing with the analytical solution for the annular plate [14], the former matrix $[S_n^{u\theta}]$ in Equation (44) is the same as Equation (36) results in the spurious eigenequation while the latter matrix $[T_n^{\mathrm{ff}}]$ results in the true eigenequation.

3.2. Discrete system for BEM

Case 1: Annular plate clamped on both the outer and inner boundaries.
When the outer and inner boundaries are both uniformly discretized into $2N$ constant elements, respectively, Equations (25) and (26) by using the complex-valued BEM can be rewritten as

$$\begin{bmatrix} U11 & U12 \\ U21 & U22 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} + \begin{bmatrix} \Theta 11 & \Theta 12 \\ \Theta 21 & \Theta 22 \end{bmatrix} \begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{45}$$

where m_1, v_1, m_2 and v_2 are the column vectors of the normal moment and effective shear force on B_1 and B_2 with a dimension $2N \times 1$, the matrices $[Uij]$ and $[\Theta ij]$ mean the influence matrices of U and Θ kernels which are obtained by collocating the field and source points on B_i and B_j with a dimension $2N \times 2N$, respectively. Similarly, Equations (27) and (28) can be rewritten as

$$\begin{bmatrix} U11_\theta & U12_\theta \\ U21_\theta & U22_\theta \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} + \begin{bmatrix} \Theta 11_\theta & \Theta 12_\theta \\ \Theta 21_\theta & \Theta 22_\theta \end{bmatrix} \begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{46}$$

where the matrices $[Uij_\theta]$ and $[\Theta ij_\theta]$ mean the influence matrices of the U_θ and Θ_θ kernels which are obtained by locating the field and source points on B_i and B_j with a dimension of $2N$ by $2N$, respectively. By assembling Equations (45) and (46) together, we have

$$[SM^{\text{cc}}] \begin{Bmatrix} v_1 \\ v_2 \\ m_1 \\ m_2 \end{Bmatrix} = \{0\} \tag{47}$$

where the superscript 'cc' denotes the clamped–clamped case and

$$[SM^{\text{cc}}] = \begin{bmatrix} U11 & U12 & \Theta 11 & \Theta 12 \\ U21 & U22 & \Theta 21 & \Theta 22 \\ U11_\theta & U11_\theta & \Theta 11_\theta & \Theta 12_\theta \\ U21_\theta & U22_\theta & \Theta 21_\theta & \Theta 22_\theta \end{bmatrix}_{8N \times 8N} \tag{48}$$

For the existence of non-trivial solution, the matrix must have a zero determinant, i.e.

$$\det[SM^{\text{cc}}] = 0 \tag{49}$$

Since the rotation symmetry is preserved for an annular boundary, the influence matrices for the discrete system are found to be the circulants. The eigenvalues ($\mu_\ell^{[U11]}$, $\mu_\ell^{[U12]}$, $\mu_\ell^{[\Theta 11]}$, $\mu_\ell^{[\Theta 12]}$, $\mu_\ell^{[U21]}$, $\mu_\ell^{[U22]}$, $\mu_\ell^{[\Theta 21]}$, $\mu_\ell^{[\Theta 22]}$, $\kappa_\ell^{[U11]}$, $\kappa_\ell^{[U12]}$, $\kappa_\ell^{[\Theta 11]}$, $\kappa_\ell^{[\Theta 12]}$, $\kappa_\ell^{[U21]}$, $\kappa_\ell^{[U22]}$, $\kappa_\ell^{[\Theta 21]}$ and $\kappa_\ell^{[\Theta 22]}$) of the sixteen influence matrices ($[U11]$, $[U12]$, $[\Theta 11]$, $[\Theta 12], \ldots, [\Theta 21_\theta]$ and $[\Theta 22_\theta]$) for the discrete system can be obtained by using the property of the circulant. By extending the relationship and employing the properties of the determinant [6, 7], we have

$$\det[SM^{\text{cc}}] = \prod_{\ell=-(N-1)}^{N} \det \begin{bmatrix} \mu_\ell^{[U11]} & \mu_\ell^{[U12]} & \mu_\ell^{[\Theta 11]} & \mu_\ell^{[\Theta 12]} \\ \mu_\ell^{[U21]} & \mu_\ell^{[U22]} & \mu_\ell^{[\Theta 21]} & \mu_\ell^{[\Theta 22]} \\ \kappa_\ell^{[U11]} & \kappa_\ell^{[U12]} & \kappa_\ell^{[\Theta 11]} & \kappa_\ell^{[\Theta 12]} \\ \kappa_\ell^{[U21]} & \kappa_\ell^{[U22]} & \kappa_\ell^{[\Theta 21]} & \kappa_\ell^{[\Theta 22]} \end{bmatrix}_{4\times 4} \tag{50}$$

By employing all the eigenvalues of the sixteen influence matrices for Equation (50), decomposition of the matrix yields

$$\det[SM^{\text{cc}}] = C_4 \prod_{\ell=-(N-1)}^{N} \det([S_\ell^{u\theta}][T_\ell^{\text{cc}}]), \quad C_4 \text{ is a constant} \tag{51}$$

where

$$[S_\ell^{u\theta}]_{4\times4}=\begin{bmatrix}(Y_\ell(\lambda a)+\mathrm{i}J_\ell(\lambda a)) & 0 & (K_\ell(\lambda a)+\mathrm{i}I_\ell(\lambda a)) & 0\\ \mathrm{i}J_\ell(\lambda b) & J_\ell(\lambda b) & \mathrm{i}I_\ell(\lambda b) & I_\ell(\lambda b)\\ (Y'_\ell(\lambda a)+\mathrm{i}J'_\ell(\lambda a)) & 0 & (K'_\ell(\lambda a)+\mathrm{i}I'_\ell(\lambda a)) & 0\\ \mathrm{i}J'_\ell(\lambda b) & J'_\ell(\lambda b) & \mathrm{i}I'_\ell(\lambda b) & I'_\ell(\lambda b)\end{bmatrix}_{4\times4} \tag{52}$$

and

$$[T_\ell^{\mathrm{cc}}]_{4\times4}=\begin{bmatrix}J_\ell(\lambda a) & J_\ell(\lambda b) & J'_\ell(\lambda a) & J'_\ell(\lambda b)\\ Y_\ell(\lambda a) & Y_\ell(\lambda b) & Y'_\ell(\lambda a) & Y'_\ell(\lambda b)\\ I_\ell(\lambda a) & I_\ell(\lambda b) & I'_\ell(\lambda a) & I'_\ell(\lambda b)\\ K_\ell(\lambda a) & K_\ell(\lambda b) & K'_\ell(\lambda a) & K'_\ell(\lambda b)\end{bmatrix}_{4\times4} \tag{53}$$

It is noted that the matrix $[T_\ell^{\mathrm{cc}}]$ denotes the matrix of true eigenequation for the C–C case and the matrix $[S_\ell^{u\theta}]$ denotes the matrix of spurious eigenequation in the u,θ formulation. Zero determinant in Equation (51) implies that the eigenequation is

$$\det([S_\ell^{u\theta}][T_\ell^{\mathrm{cc}}])=0,\quad \ell=0,\pm1,\pm2,\ldots,\pm(N-1),N \tag{54}$$

After comparing with the analytical solution for the annular plate [14], the former matrix $[S_\ell^{u\theta}]$ in Equation (54) results in the spurious eigenequation while the latter matrix $[T_\ell^{\mathrm{cc}}]$ results in the true eigenequation. The results of Equation (53) in the discrete system match well with the former one in the continuous system.

Case 2: Annular plate simply supported on both the outer and inner boundaries. Following the same procedure of case 1, we have

$$[SM^{\mathrm{ss}}]=\begin{bmatrix}U11 & U12 & M11 & M12\\ U21 & U22 & M21 & M22\\ U11_\theta & U11_\theta & M11_\theta & M12_\theta\\ U21_\theta & U22_\theta & M21_\theta & M22_\theta\end{bmatrix}_{8N\times8N} \tag{55}$$

where the superscript ‘ss’ denotes the simply supported–simply supported case. Based on the theory of circulant, we have

$$\det[SM^{\mathrm{ss}}]=C_5\prod_{\ell=-(N-1)}^{N}\det([S_\ell^{u\theta}][T_\ell^{\mathrm{ss}}]),\quad C_5 \text{ is a constant} \tag{56}$$

where

$$[T_\ell^{\mathrm{ss}}]_{4\times4}=\begin{bmatrix}J_\ell(\lambda a) & J_\ell(\lambda b) & \alpha_\ell^J(\lambda a) & \alpha_\ell^J(\lambda b)\\ Y_\ell(\lambda a) & Y_\ell(\lambda b) & \alpha_\ell^Y(\lambda a) & \alpha_\ell^Y(\lambda b)\\ I_\ell(\lambda a) & I_\ell(\lambda b) & \alpha_\ell^I(\lambda a) & \alpha_\ell^I(\lambda b)\\ K_\ell(\lambda a) & K_\ell(\lambda b) & \alpha_\ell^K(\lambda a) & \alpha_\ell^K(\lambda b)\end{bmatrix}_{4\times4} \tag{57}$$

It is noted that the matrix $[T_\ell^{\mathrm{ss}}]$ denotes the matrix of true eigenequation for the simply supported–simply supported case. Zero determinant in Equation (56) implies that the eigenequation is

$$\det([S_\ell^{u\theta}][T_\ell^{\mathrm{ss}}]) = 0, \quad \ell = 0, \pm 1, \pm 2, \ldots, \pm(N-1), N \tag{58}$$

The results of Equation (57) in the discrete system match well with the former one in the continuous system.

Case 3: Annular plate free on both the outer and inner boundaries.
Similarly, we have

$$[SM^{\mathrm{ff}}] = \begin{bmatrix} M11 & M12 & V11 & V12 \\ M21 & M22 & V21 & V22 \\ M11_\theta & M11_\theta & V11_\theta & V12_\theta \\ M21_\theta & M22_\theta & V21_\theta & V22_\theta \end{bmatrix}_{8N\times 8N} \tag{59}$$

where the superscript 'ff' denotes the free–free case. Also, we have

$$\det[SM^{\mathrm{ff}}] = C_6 \prod_{\ell=-(N-1)}^{N} \det([S_\ell^{u\theta}][T_\ell^{\mathrm{ff}}]), \quad C_6 \text{ is a constant} \tag{60}$$

where

$$[T_\ell^{\mathrm{ff}}] = \begin{bmatrix} \alpha_\ell^J(\lambda a) & \alpha_\ell^J(\lambda b) & \beta_\ell^J(\lambda a) + \dfrac{(1-v)}{a}\gamma_\ell^J(\lambda a) & \beta_\ell^J(\lambda b) + \dfrac{(1-v)}{b}\gamma_\ell^J(\lambda b) \\ \alpha_\ell^Y(\lambda a) & \alpha_\ell^Y(\lambda b) & \beta_\ell^Y(\lambda a) + \dfrac{(1-v)}{a}\gamma_\ell^Y(\lambda a) & \beta_\ell^Y(\lambda b) + \dfrac{(1-v)}{b}\gamma_\ell^Y(\lambda b) \\ \alpha_\ell^I(\lambda a) & \alpha_\ell^I(\lambda b) & \beta_\ell^I(\lambda a) + \dfrac{(1-v)}{a}\gamma_\ell^I(\lambda a) & \beta_\ell^I(\lambda b) + \dfrac{(1-v)}{b}\gamma_\ell^I(\lambda b) \\ \alpha_\ell^K(\lambda a) & \alpha_\ell^K(\lambda b) & \beta_\ell^K(\lambda a) + \dfrac{(1-v)}{a}\gamma_\ell^K(\lambda a) & \beta_\ell^K(\lambda b) + \dfrac{(1-v)}{b}\gamma_\ell^K(\lambda b) \end{bmatrix}_{4\times 4} \tag{61}$$

It is noted that the matrix $[T_\ell^{\mathrm{ff}}]$ denotes the matrix of true eigenequation for the F–F case. Zero determinant in Equation (60) implies that the eigenequation is

$$\det([S_\ell^{u\theta}][T_\ell^{\mathrm{ff}}]) = 0, \quad \ell = 0, \pm 1, \pm 2, \ldots, \pm(N-1), N \tag{62}$$

The results of Equation (61) in the discrete system match well with the former one in the continuous system.

The proof can be easily extended to problems subject to the different combinations of boundary conditions on the outer boundary and inner boundary. All the results for the annular plate subject to different boundary conditions are shown in Table I.

Table I. True eigenequations for the annular plate.

Cases	True eigenequation $[T_n]$
C–C	$\begin{bmatrix} J_n(\lambda a) & J_n(\lambda b) & J'_n(\lambda a) & J'_n(\lambda b) \\ Y_n(\lambda a) & Y_n(\lambda b) & Y'_n(\lambda a) & Y'_n(\lambda b) \\ I_n(\lambda a) & I_n(\lambda b) & I'_n(\lambda a) & I'_n(\lambda b) \\ K_n(\lambda a) & K_n(\lambda b) & K'_n(\lambda a) & K'_n(\lambda b) \end{bmatrix}$
S–S	$\begin{bmatrix} J_n(\lambda a) & J_n(\lambda b) & \alpha_n^J(\lambda a) & \alpha_n^J(\lambda b) \\ Y_n(\lambda a) & Y_n(\lambda b) & \alpha_n^Y(\lambda a) & \alpha_n^Y(\lambda b) \\ I_n(\lambda a) & I_n(\lambda b) & \alpha_n^I(\lambda a) & \alpha_n^I(\lambda b) \\ K_n(\lambda a) & K_n(\lambda b) & \alpha_n^K(\lambda a) & \alpha_n^K(\lambda b) \end{bmatrix}$
F–F	$\begin{bmatrix} \alpha_n^J(\lambda a) & \alpha_n^J(\lambda b) & \beta_n^J(\lambda a)+\frac{1-\nu}{b}\gamma_n^J(\lambda a) & \beta_n^J(\lambda b)+\frac{1-\nu}{b}\gamma_n^J(\lambda b) \\ \alpha_n^Y(\lambda a) & \alpha_n^Y(\lambda b) & \beta_n^Y(\lambda a)+\frac{1-\nu}{b}\gamma_n^Y(\lambda a) & \beta_n^Y(\lambda b)+\frac{1-\nu}{b}\gamma_n^Y(\lambda b) \\ \alpha_n^I(\lambda a) & \alpha_n^I(\lambda b) & \beta_n^I(\lambda a)+\frac{1-\nu}{b}\gamma_n^I(\lambda a) & \beta_n^I(\lambda b)+\frac{1-\nu}{b}\gamma_n^I(\lambda b) \\ \alpha_n^K(\lambda a) & \alpha_n^K(\lambda b) & \beta_n^K(\lambda a)+\frac{1-\nu}{b}\gamma_n^K(\lambda a) & \beta_n^K(\lambda b)+\frac{1-\nu}{b}\gamma_n^K(\lambda b) \end{bmatrix}$

3.3. Study of the spurious eigenequation in the [S] matrix

After comparing Equation (54) with Equations (58) and (62) in the discrete system or comparing the results of Equation (38) with Equations (41) and (44) in the continuous system for the annular plate, the same spurious eigenequation ($[S_n^{u\theta}]=0$) is embedded in the u,θ formulation no matter what the boundary condition is. By using the cofactor of the matrix $[S_n^{u\theta}]$ to simplify the zero determinant of Equation (36) for the spurious eigenequation, we have

$$\det[S_n^{u\theta}]_{4\times 4}=\det([Sa_n^{u\theta}][Sb_n^{u\theta}]) \tag{63}$$

where

$$[Sa_n^{u\theta}]=\begin{bmatrix} (Y_n(\lambda a)+\mathrm{i}J_n(\lambda a)) & (K_n(\lambda a)+\mathrm{i}I_n(\lambda a)) \\ (Y'_n(\lambda a)+\mathrm{i}J'_n(\lambda a)) & (K'_n(\lambda a)+\mathrm{i}I'_n(\lambda a)) \end{bmatrix}_{2\times 2} \tag{64}$$

and

$$[Sb_n^{u\theta}]=\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ J'_n(\lambda b) & I'_n(\lambda b) \end{bmatrix}_{2\times 2} \tag{65}$$

It is found that the determinant of the former matrix $[Sa_n^{u\theta}]$ in Equation (64) is never zero for any λ. The spurious eigenequation is the zero determinant of the matrix $[Sb_n^{u\theta}]$ in Equation (65)

Table II. Spurious eigenequations for the annular plate.

Formulation	$[Sb_n]$	Boundary condition of the simply connected plate
u, θ formulation	$\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \lambda(J_n'(\lambda b)) & \lambda(I_n'(\lambda b)) \end{bmatrix}$	$u=0,\ \theta=0$
u, m formulation	$\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \end{bmatrix}$	$u=0,\ m=0$
u, v formulation	$\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \left[\beta_n^J(\lambda b)+\dfrac{(1-\nu)}{b}\gamma_n^J(\lambda b)\right] & \left[\beta_n^I(\lambda b)+\dfrac{(1-\nu)}{b}\gamma_n^I(\lambda b)\right] \end{bmatrix}$	$u=0,\ v=0$
θ, m formulation	$\begin{bmatrix} \lambda J_n'(\lambda b) & \lambda I_n'(\lambda b) \\ \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \end{bmatrix}$	$\theta=0,\ m=0$
θ, v formulation	$\begin{bmatrix} \lambda J_n'(\lambda b) & \lambda I_n'(\lambda b) \\ \left[\beta_n^J(\lambda b)+\dfrac{(1-\nu)}{b}\gamma_n^J(\lambda b)\right] & \left[\beta_n^I(\lambda b)+\dfrac{(1-\nu)}{b}\gamma_n^I(\lambda b)\right] \end{bmatrix}$	$\theta=0,\ v=0$
m, v formulation	$\begin{bmatrix} \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \\ \left[\beta_n^J(\lambda b)+\dfrac{(1-\nu)}{b}\gamma_n^J(\lambda b)\right] & \left[\beta_n^I(\lambda b)+\dfrac{(1-\nu)}{b}\gamma_n^I(\lambda b)\right] \end{bmatrix}$	$m=0,\ v=0$

which only relates to the inner radius b. It is interesting that the zero determinant of the $[Sb_n^{u\theta}]$ in the u, θ formulation results in the true eigenequation of a clamped plate with a radius b. The spurious eigenvalues parasitizing in the u, θ formulation depend on the radius b which is the inner circle of the annular domain. In fact, the multiply connected problem can be seen as a superposition of two problems, one is an interior problem with the boundary, B_2, and the other is an exterior problem with the boundary, B_1. The source which causes the appearance of the spurious eigenvalues stems from the exterior problem with the inner boundary even though the complex-valued kernels are employed. This finding for the plate is similar to that of membrane vibration and acoustics [10, 11].

Since any two equations in the plate formulation (Equations (17)–(20)) can be chosen, $6(C_2^4)$ options of the formulation can be considered. All the results of the spurious eigenequation are shown in Table II. It is found that spurious eigenequation for the annular case is the true eigenequation of the circular plate with radius b. The occurrence of spurious eigenequation only depends on the adopted formulation instead of the specified boundary condition. True eigenequation depends on the specified boundary condition instead of the formulation.

4. TREATMENT OF THE SPURIOUS EIGENVALUES FOR AN ANNULAR PLATE USING BIEM AND BEM

4.1. SVD updating technique

In the discrete system, the approach to detect the true eigensolution is the criterion of satisfying all Equations (17)–(20) at the same time by using the complex-valued BEM. After rearranging the terms of Equations (17) and (18), we have

$$[SM_1^{\mathrm{cc}}]=\begin{bmatrix} U11 & U12 & \Theta 11 & \Theta 12 \\ U21 & U22 & \Theta 21 & \Theta 22 \\ U11_\theta & U12_\theta & \Theta 11_\theta & \Theta 12_\theta \\ U21_\theta & U22_\theta & \Theta 21_\theta & \Theta 22_\theta \end{bmatrix} \tag{66}$$

Similarly, Equations (19) and (20) yield

$$[SM_2^{\mathrm{cc}}]=\begin{bmatrix} U11_m & U12_m & \Theta 11_m & \Theta 12_m \\ U21_m & U22_m & \Theta 21_m & \Theta 22_m \\ U11_v & U12_v & \Theta 11_v & \Theta 12_v \\ U21_v & U22_v & \Theta 21_v & \Theta 22_v \end{bmatrix} \tag{67}$$

By using the SVD technique of updating term for the clamped case [11], we have

$$[C]\begin{Bmatrix} v_1 \\ v_2 \\ m_1 \\ m_2 \end{Bmatrix}=\{0\} \tag{68}$$

where

$$[C]=\begin{bmatrix} SM_1^{\mathrm{cc}} \\ SM_2^{\mathrm{cc}} \end{bmatrix}_{16N\times 8N} \tag{69}$$

Since the eigenequation is non-trivial, the rank of the matrix $[C]$ must be smaller than $8N$, the $8N$ singular values for the matrix $[C]$ must be zero at least. We can find that the determinant of the matrix $[C]^{\mathrm{T}}[C]$ can be decomposed into the summation of the square determinant in the C_4^8 matrices. The only possibility for the zero determinant of the matrix $[C]^{\mathrm{T}}[C]$ occurs when the C_4^8 terms are all zeros at the same time. After a careful check for all the matrices, we find that the true eigenequation $[T_\ell^{\mathrm{cc}}]$ is simultaneously embedded in the C_4^8 matrices. This indicates that only the true eigenequation of the clamped–clamped annular plate is sorted out in the SVD updating matrix since the true eigenequation is simultaneously embedded in the six complex-valued formulations. The result matches well with the former one in the discrete system, respectively.

4.2. Burton and Miller method

By combining Equations (66) and (67) with an imaginary number in the complex-valued BEM, we have

$$[[SM_1^{\mathrm{cc}}]+\mathrm{i}[SM_2^{\mathrm{cc}}]]\begin{Bmatrix} v_1 \\ v_2 \\ m_1 \\ m_2 \end{Bmatrix}=\{0\} \tag{70}$$

By using the circulant and the decomposition technique, the determinant of the $[SM_1^{\mathrm{cc}}]+\mathrm{i}[SM_2^{\mathrm{cc}}]$ is obtained

$$\det[[SM_1^{\mathrm{cc}}]+\mathrm{i}[SM_2^{\mathrm{cc}}]]=\prod_{\ell=-(N-1)}^{N}\det([[S_\ell^{u\theta}]+\mathrm{i}[S_\ell^{mv}]][T_\ell^{\mathrm{cc}}]) \tag{71}$$

Since the term $[S_\ell^{u\theta}]+\mathrm{i}[S_\ell^{mv}]$ is never zero for any λ, we can obtain the true eigenvalues by using the complex-valued BEM in conjunction with the Burton and Miller concept. Nevertheless, if we combine the u,θ and m,v formulations or u,v and θ,m formulations, the method fails. The reason is that the u,v and θ,m formulation have the same spurious eigenequation. Only the combination of u,m and θ,v complex-valued formulations can obtain the true eigenvalues. All the explicit forms of the $[S_\ell^{u\theta}]+\mathrm{i}[S_\ell^{mv}]$ are shown in Table III by using the complex-valued BEM. Since any two equations in the complex-valued formulation result in the spurious eigenvalues, we can reconstruct the independent equation by employing the Burton and Miller concept. When we choose the appropriate combination, the Burton and Miller method works well.

4.3. CHIEF method

By adding the point with a radius ρ for the null-field equation to solve the eigenproblem of annular plate, we have two choices for the location of CHIEF point ($\rho<b$) or CHEEF point ($a<\rho$). If the CHEEF point locates on the outer the domain ($a<\rho$), the CHEEF method fails [11]. By moving the field point x to be outside the domain ($\rho<b$) for CHIEF points, we have

$$\begin{bmatrix} UC1 & UC2 & \Theta C1 & \Theta C2 \\ UC1_\theta & UC2_\theta & \Theta C1_\theta & \Theta C2_\theta \end{bmatrix}_{2N_C\times 8N}\begin{Bmatrix} v_1 \\ v_2 \\ m_1 \\ m_2 \end{Bmatrix}_{8N\times 1}=\{0\}_{2N_c\times 1} \tag{72}$$

where the index C denotes the CHIEF point in the null-field integral equation and the matrix dimension $N_c\,(\geqslant 1)$ indicates the number of additional CHIEF points. The submatrices in Equation (72) can be obtained by adding the influence row vectors resulted from the U,Θ,U_θ and Θ_θ kernels due to the CHIEF point. Combining Equations (66) and (72) together to obtain

Table III. The terms of $[Sb_n^1]+\mathrm{i}[Sb_n^2]$ for the annular plate by using the complex-valued BEM in conjunction the Burton and Miller method.

	$[Sb_n^1]+\mathrm{i}[Sb_n^2]$
u,θ formulation	$\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \lambda(J_n'(\lambda b)) & \lambda(I_n'(\lambda b)) \end{bmatrix} + \mathrm{i}\begin{bmatrix} \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \\ \beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b) & \beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b) \end{bmatrix}$
u,m formulation	$\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \end{bmatrix} + \mathrm{i}\begin{bmatrix} \lambda J_n'(\lambda b) & \lambda I_n'(\lambda b) \\ \beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b) & \beta_n^I(\lambda b)+\frac{(1+\nu)}{b}\gamma_n^I(\lambda b) \end{bmatrix}$
u,v formulation	$\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b) & \beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b) \end{bmatrix} + \mathrm{i}\begin{bmatrix} \lambda J_n'(\lambda b) & \lambda I_n'(\lambda b) \\ \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \end{bmatrix}$
θ,m formulation	$\begin{bmatrix} \lambda J_n'(\lambda b) & \lambda I_n'(\lambda b) \\ \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \end{bmatrix} + \mathrm{i}\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b) & \beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b) \end{bmatrix}$
θ,v formulation	$\begin{bmatrix} \lambda J_n'(\lambda b) & \lambda I_n'(\lambda b) \\ \beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b) & \beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b) \end{bmatrix} + \mathrm{i}\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \end{bmatrix}$
m,v formulation	$\begin{bmatrix} \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \\ \beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b) & \beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b) \end{bmatrix} + \mathrm{i}\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \lambda(J_n'(\lambda b)) & \lambda(I_n'(\lambda b)) \end{bmatrix}$

the overdetermined system, we have

$$[C^*]\begin{Bmatrix} v_1 \\ v_2 \\ m_1 \\ m_2 \end{Bmatrix}_{8N\times 1} = \{0\}_{(8N+2N_C)\times 1} \tag{73}$$

where

$$[C^*]=\left[\begin{array}{cccc} U11 & U12 & \Theta 11 & \Theta 12 \\ U21 & U22 & \Theta 21 & \Theta 22 \\ U11_\theta & U12_\theta & \Theta 11_\theta & \Theta 12_\theta \\ U21_\theta & U22_\theta & \Theta 21_\theta & \Theta 22_\theta \\ \hline UC1 & UC2 & \Theta C1 & \Theta C2 \\ UC1_\theta & UC2_\theta & \Theta C1_\theta & \Theta C2_\theta \end{array}\right]_{(8N+2N_C)\times 8N} \tag{74}$$

Therefore, an overdetermined system is obtained to filter out the spurious eigenvalues.

5. NUMERICAL RESULTS AND DISCUSSIONS

An annular plate with the outer radius of 1 m ($a = 1$ m) and the inner radius of 0.5 m ($b = 0.5$ m) of B_1 and B_2, respectively, and the Poisson ratio $v = 1/3$ is considered. The outer and inner boundaries are both uniformly discretized into ten constant elements, respectively.

Figures 1–3 show the determinant of $[SM]$ versus the frequency parameter λ for the three cases of annular plate using the six complex-valued formulations. Both the true and spurious eigenvalues occur simultaneously even though the complex-valued BEM is employed. After comparing with (a)–(f) results for each figure, the same true eigenvalues are obtained no matter what the adopted formulation is. It reconfirms that the true eigenvalues depends on the specified boundary condition instead of the formulation. After selecting the formulation (e.g. u, θ formulation), the spurious eigenvalues (6.392, 9.222 and 11.810) occur at the positions which satisfy the spurious eigenequation $\det|[S_n^{u\theta}]| = 0$ in Equation (36) as shown in Figures 1(a), 2(a) and 3(a). In order to distinguish the spurious eigenvalues, Figure 4(a)–(c) and 4(d)–(f) show the determinant of $[SM]$ versus λ using the same formulation (4(a)–(c) for u, θ formulation; 4(d)–(f) for u, m formulation) to solve the plates subject to different boundary conditions. The numerical results reconfirm that the occurrence of spurious eigenvalues only depends on the formulation instead of the specified boundary condition.

The true eigenvalues (6.392, 9.222 and 11.810) for the circular clamped plate in Figure 5(a) with a radius $b = 0.5$ m appears at the same positions of the spurious eigenvalues in Figures 1(a), 2(a) and 3(a) when using the u, θ complex-valued BEM for the annular plate. In other words, the spurious eigenvalues embedded in each (C_2^4) formulation for the annular plate are corresponding to the associated true eigenvalues of the inner circular plate as shown in Figure 5(a)–(f).

Treatment of the spurious eigenvalues

Figure 6(a) and (d) show the determinant of the $[C]^{\mathrm{T}}[C]$ versus λ for the F–F annular plate using the complex-valued formulations in conjunction with the SVD technique of updating term. It is found that all the spurious eigenvalues are filtered out and only the true eigenvalues appear. Figure 6(b) and (e) show the determinant of the $[SM]$ versus λ for the F–F annular plate using the six complex-valued formulations in conjunction with the Burton and Miller concept. Only the combination of u, m and θ, v formulation can obtain the true eigenvalues in Figure 6(b) and (e) as predicted in Table III, since $\det([Sb_n^{um}] + \mathrm{i}[Sb_n^{\theta v}])$ cannot be zero. Figure 6(c) show the minimum singular value σ_1 of the $[C^*]$ versus λ for the F–F annular plate by using the u, θ formulations in conjunction with the two CHIEF points. For Figure 6(c), the CHIEF points locate at $(0.285, \pi/4)$ and $(0.275, 29\pi/36)$, where the angle between the two selected points is $5\pi/9$. Similarly, Figure 6(f) show the minimum singular value σ_1 of the $[C^*]$ versus λ for the F–F annular plate by using the u, m formulation in conjunction with the two CHIEF points. The CHIEF points locate at $(0.30, \pi/4)$ and $(0.28, 29\pi/36)$, where the angle between the two selected points is $5\pi/9$. Good agreement is made by using the CHIEF method. Only the true eigenvalues are obtained.

In general, all the cases result in the same spurious eigenvalues, once the formulation is adopted no matter what the boundary condition is specified. All the numerical data of the true eigenvalues are summarized in Table IV(a)–(c), and the eigenvalues agree well with the data in References [14–16] to match our solution. However, the obtained eigenvalues according to the Leissa's eigenequation are not consistent to those in his book. The possible explanation is

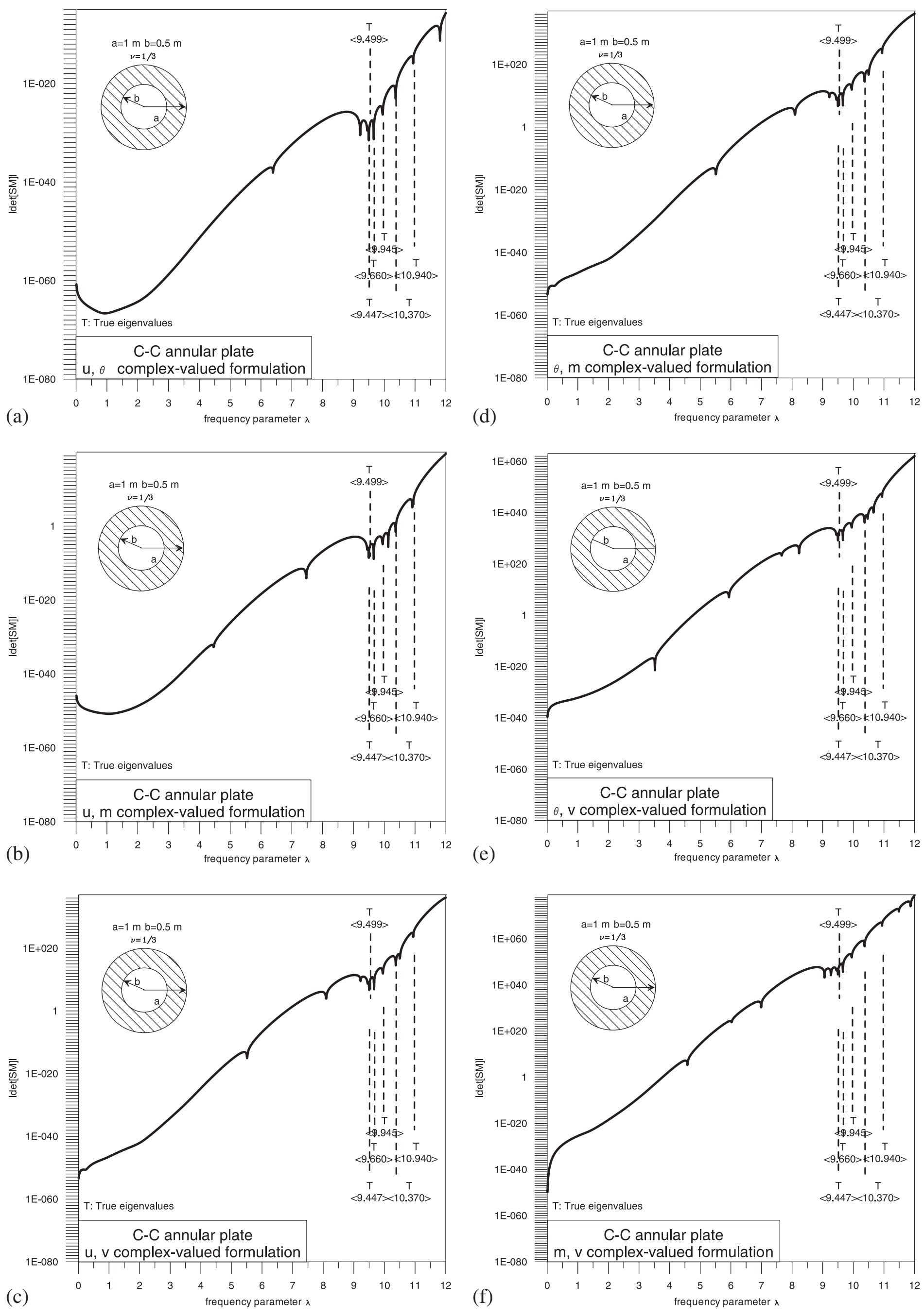

Figure 1. The determinant of the $[SM^{cc}]$ versus the frequency parameter λ for the C–C annular plate using the six complex-valued formulations.

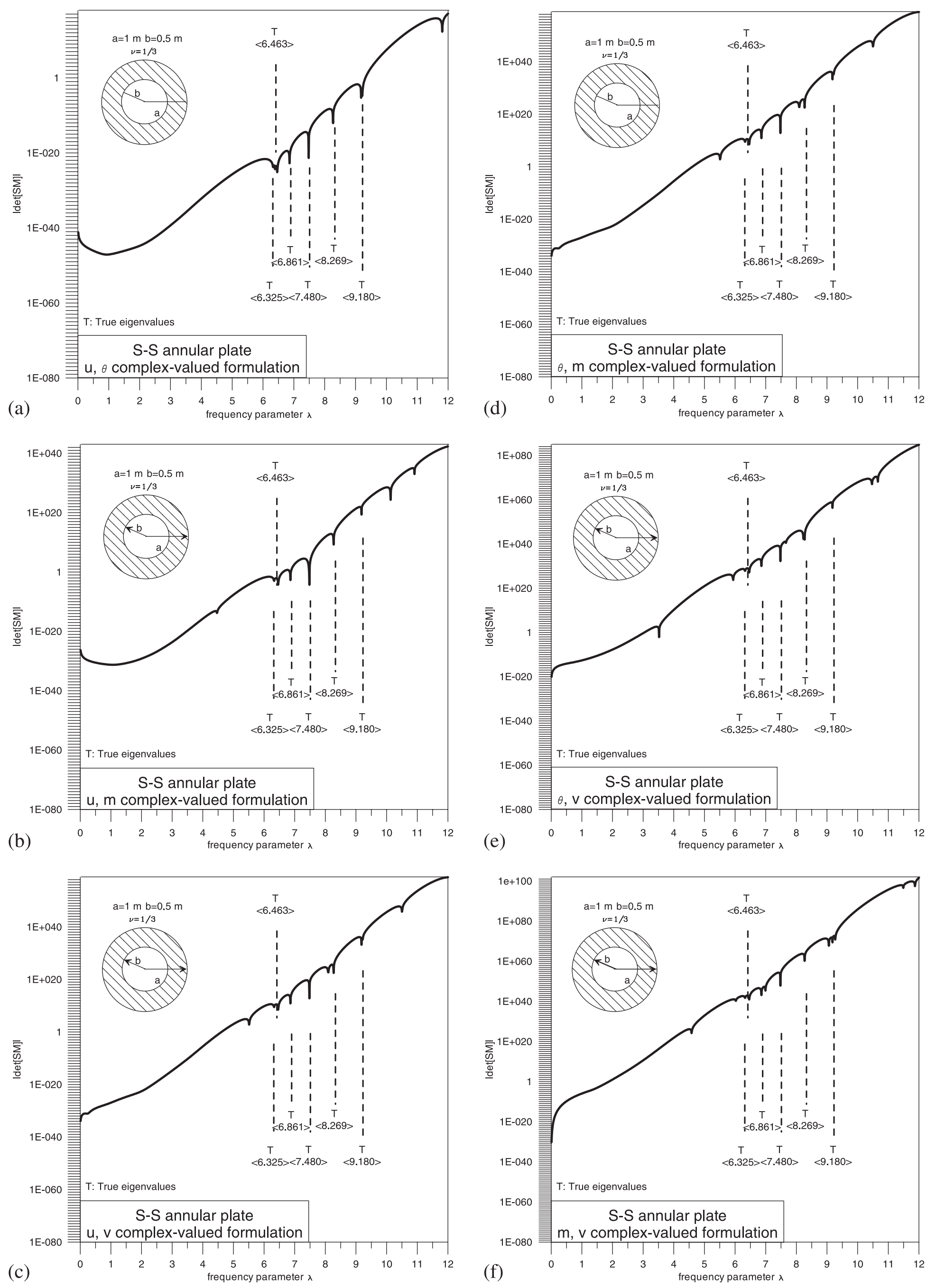

Figure 2. The determinant of the $[SM^{ss}]$ versus the frequency parameter λ for the S–S annular plate using the six complex-valued formulations.

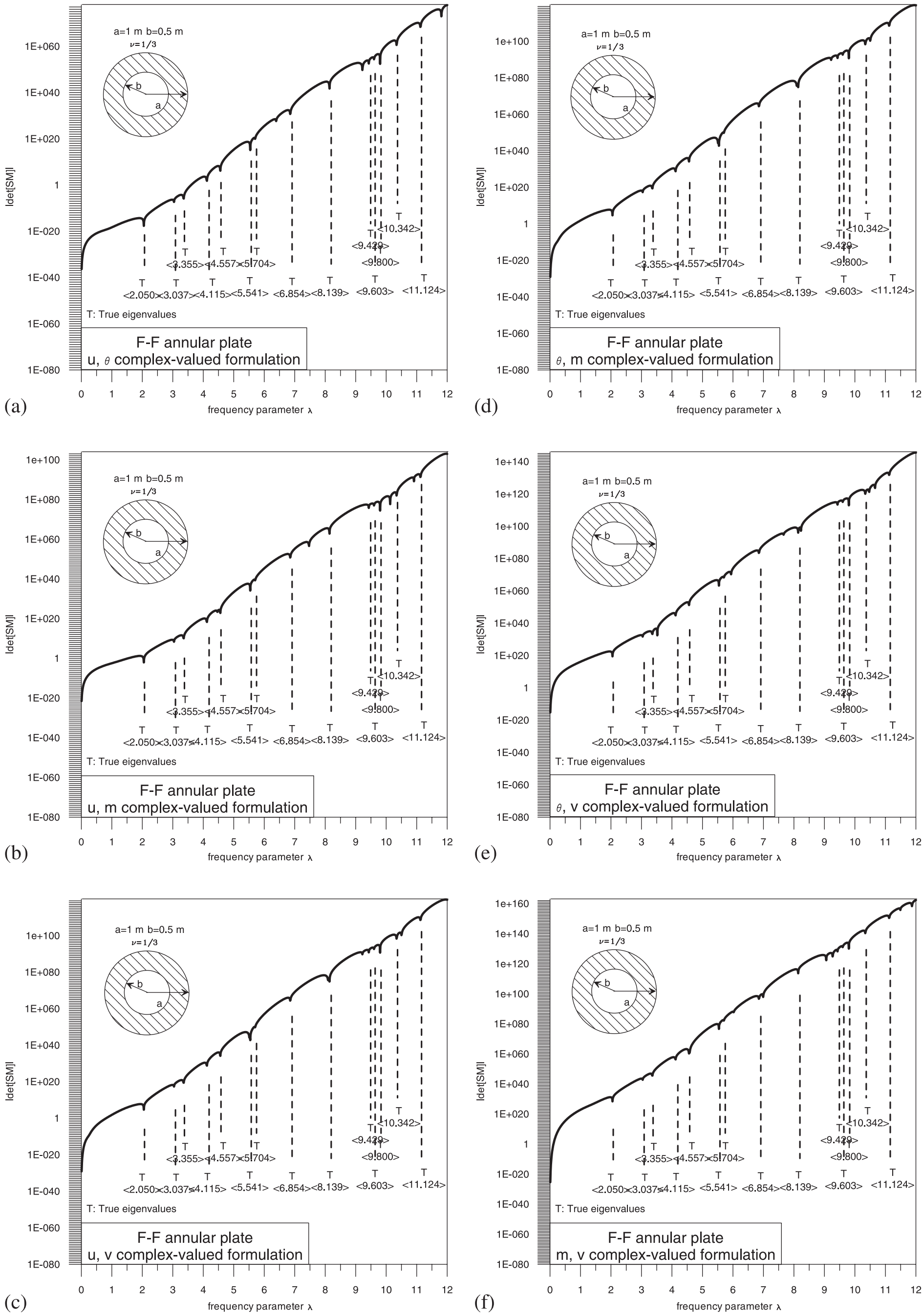

Figure 3. The determinant of the $[SM^{\mathrm{ff}}]$ versus the frequency parameter λ for the F–F annular plate using the six complex-valued formulations.

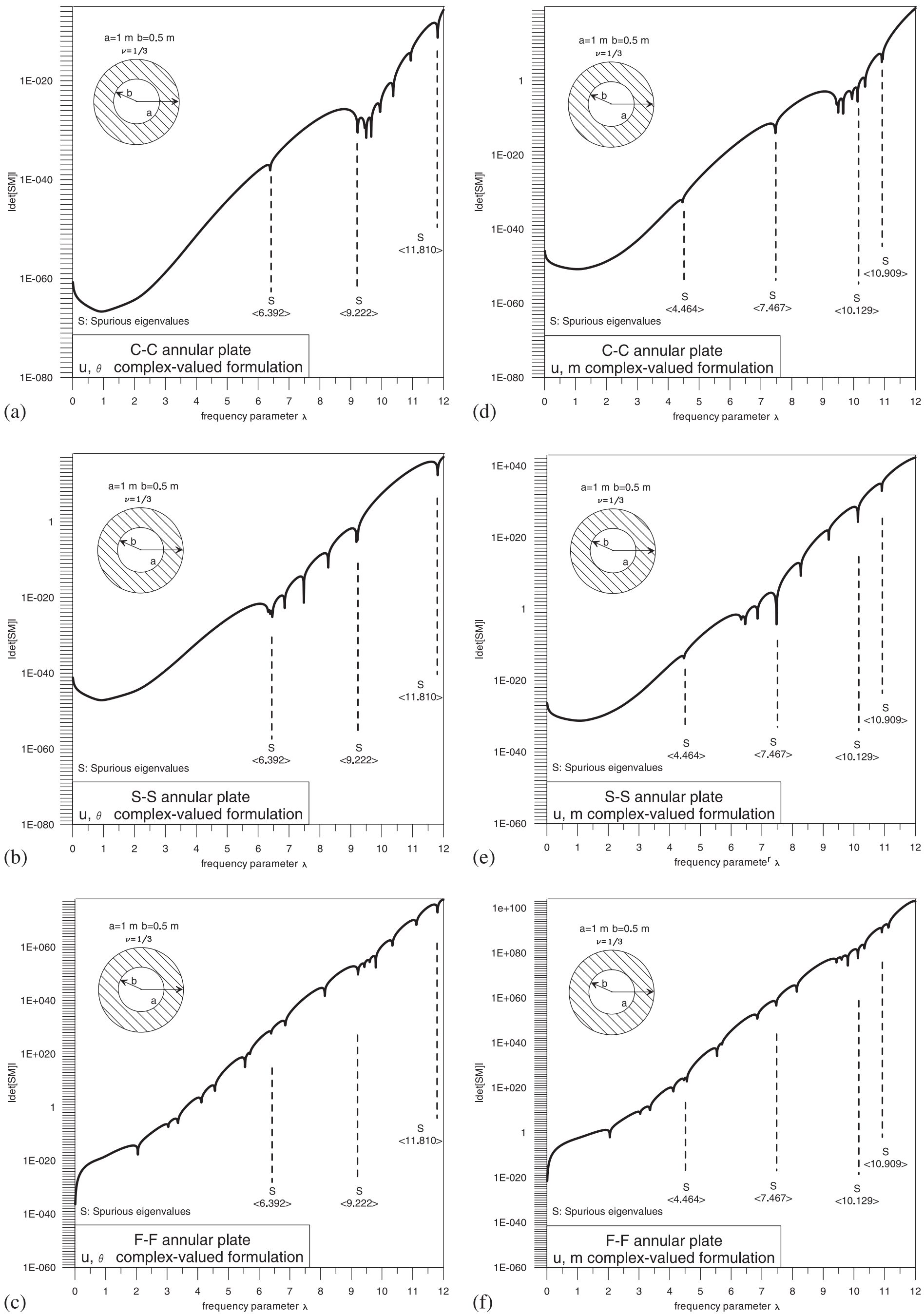

Figure 4. The determinant of the $[SM]$ versus the frequency parameter λ using the complex-valued formulation (u, θ or u, m formulation) to solve plates subject to different boundary conditions.

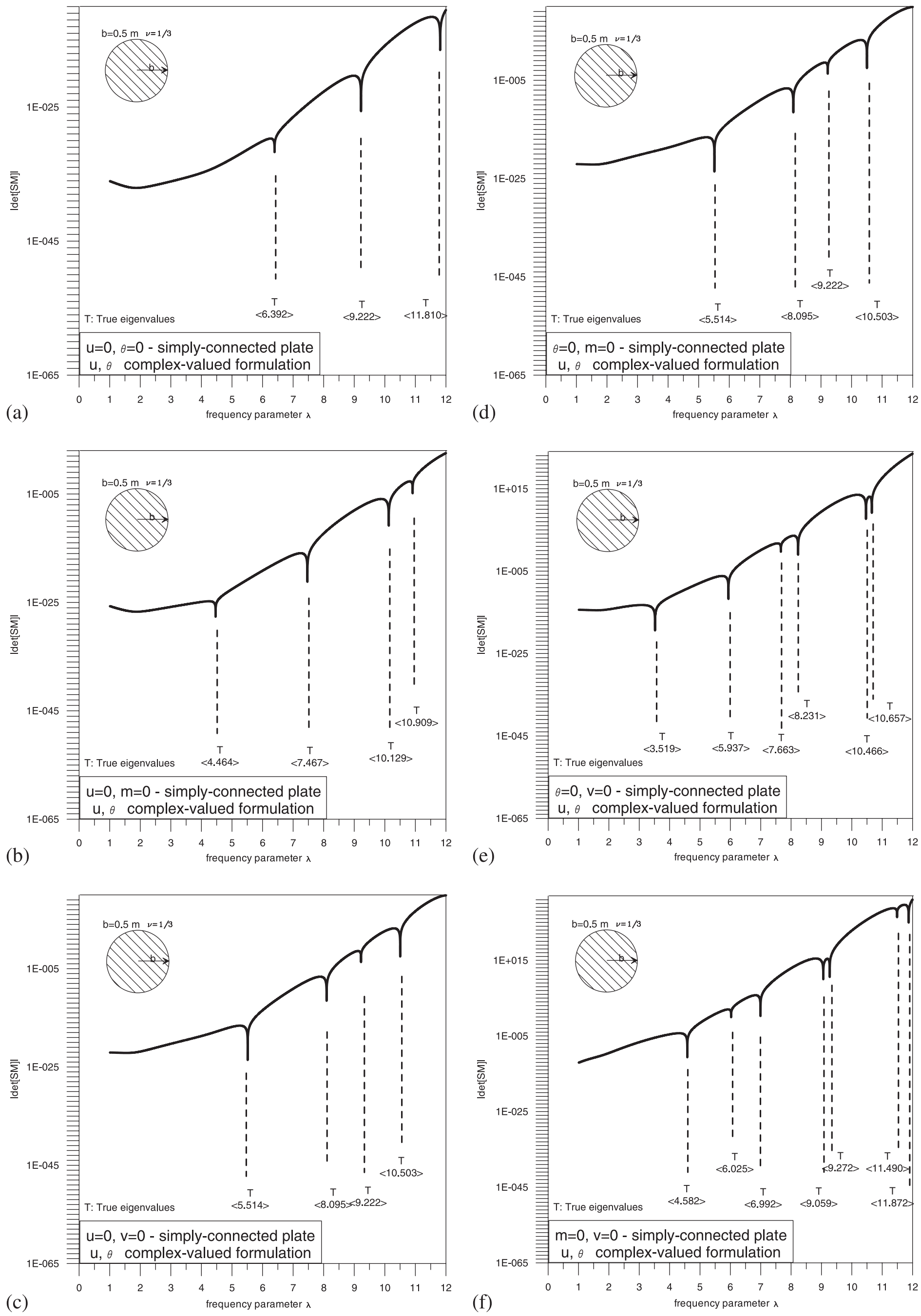

Figure 5. The determinant of the $[SM]$ versus the frequency parameter λ using the complex-valued formulation to solve plates subject to different boundary conditions for the simply connected plate with a radius b.

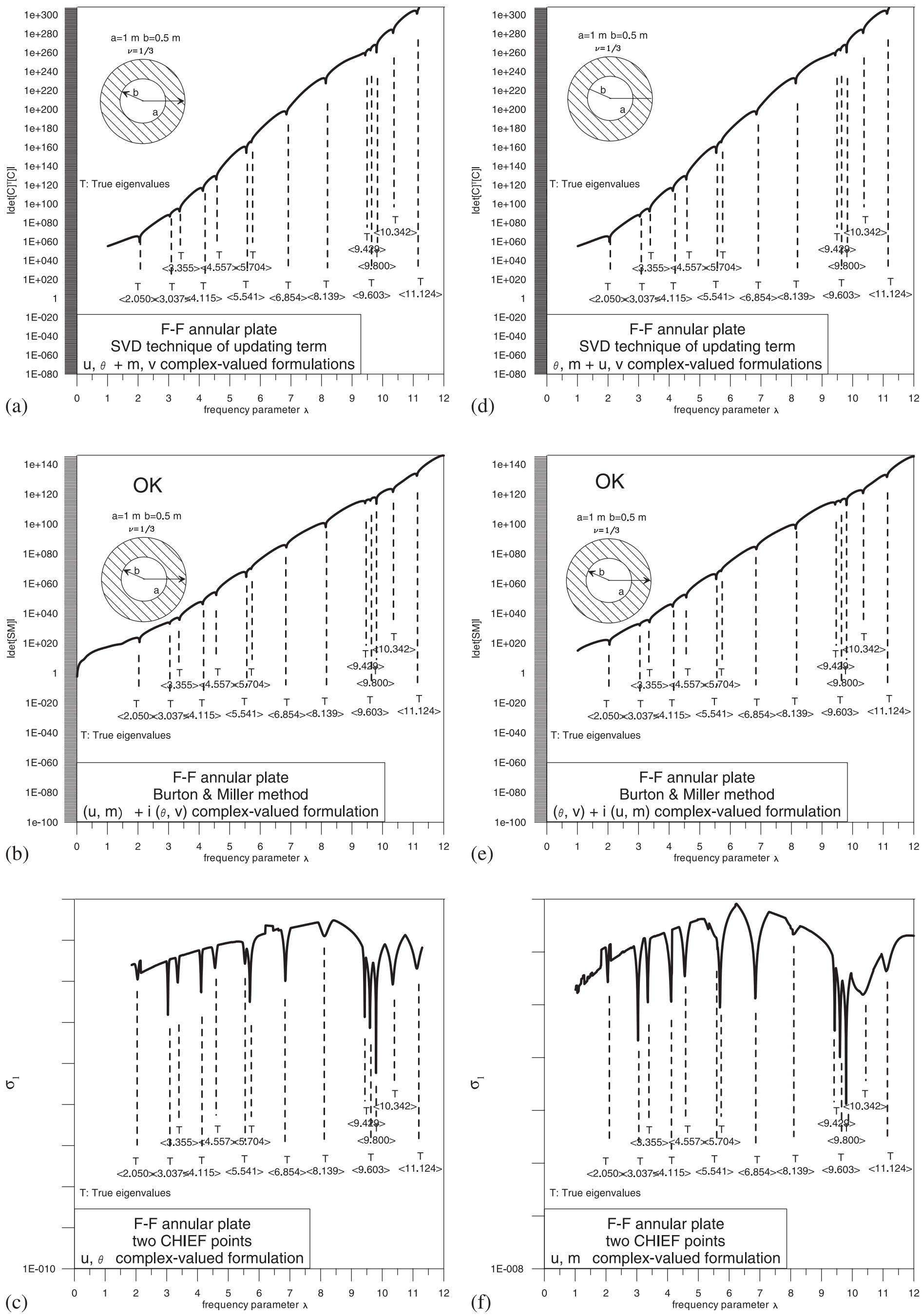

Figure 6. Three alternatives (SVD updating term, the Burton and Miller method and CHIEF method) for the F–F annular plate using the complex-valued formulations.

Table IV. True eigenvalues (λ) for the annular plate ($a=1$, $b=0.5$ and $\nu=1/3$).

	λ for values of n of					
m	0	1	2	3	4	5
(a) True eigenvalues (λ) for the C–C case ($a=1$, $b=0.5$)						
0	9.447	9.499	9.660	9.945	10.370	10.940
1	15.694	15.739	15.873	16.098	16.415	16.827
(b) True eigenvalues (λ) for the S–S case ($a=1$, $b=0.5$, $\nu=1/3$)						
0	6.325	6.463	6.861	7.480	8.269	9.180
1	12.592	12.669	12.895	13.265	13.767	14.391
(c) True eigenvalues (λ) for the F–F case ($a=1$, $b=0.5$, $\nu=1/3$)						
0	3.037	4.115	2.050	3.355	4.557	5.704
1	9.603	9.800	5.541	6.854	8.139	9.429

n is the number of nodal diameters and m the number of nodal circles, not including the boundary circle.

that the eigenequations in the Leissa's book [14] for some cases was wrongly typed. Recently, we have found the new edition of Leissa's book [17] and the typing error has been corrected to match our solution.

6. CONCLUSIONS

A complex-valued boundary integral equation has been formulated for the free vibration of annular plate. The true and spurious eigenequations were derived analytically by using the Fourier series, degenerate kernels and circulants in both the continuous system (BIEM) and discrete system (BEM) while the eigenvalues were determined numerically. Since either two equations in the plate formulation (4 equations) can be chosen, C_2^4 (6) options can be considered. The occurrence of spurious eigenequation only depends on the formulation instead of the specified boundary condition, while the true eigenequation is independent of the formulation and is relevant to the specified boundary condition. It is interesting that the spurious eigenequation of annular plate eigenproblem by using the u, θ formulation is found to be the true eigenequation of clamped circular plate with a radius b which is the inner radius of the annular plate. Several examples of plates were illustrated to check the validity of the present formulations. Three alternatives (SVD updating technique, Burton and Miller method and the CHIEF method) were adopted to suppress the occurrence of the spurious eigenvalues for the C−C, S−S and F−F annular plates in the complex-valued BEM. Although the annular case lacks generality, it leads significant insight into the occurring mechanism of true and spurious eigenequation for multiply connected eigenproblems. It is also a great help to the researchers who may require analytical explanation for the reason why spurious eigenvalues appear for the multiply connected problems. The same algorithm in the discrete system can be applied to solve arbitrary-shaped plates numerically without any difficulty; however, analytical derivation in the continuous and discrete systems cannot be done as the annular case can.

APPENDIX A: LIST OF THE COEFFICIENTS

$$\alpha_m^Y(\lambda a) = \lambda^2 Y_m''(\lambda a) + v\left[\frac{1}{a}\lambda Y_m'(\lambda a) - \left(\frac{m}{a}\right)^2 Y_m(\lambda a)\right]$$

$$\alpha_m^J(\lambda a) = \lambda^2 J_m''(\lambda a) + v\left[\frac{1}{a}\lambda J_m'(\lambda a) - \left(\frac{m}{a}\right)^2 J_m(\lambda a)\right]$$

$$\alpha_m^K(\lambda a) = \lambda^2 K_m''(\lambda a) + v\left[\frac{1}{a}\lambda K_m'(\lambda a) - \left(\frac{m}{a}\right)^2 K_m(\lambda a)\right]$$

$$\alpha_m^I(\lambda a) = \lambda^2 I_m''(\lambda a) + v\left[\frac{1}{a}\lambda I_m'(\lambda a) - \left(\frac{m}{a}\right)^2 I_m(\lambda a)\right]$$

$$\beta_m^Y(\lambda a) = \lambda^3 Y_m'''(\lambda a) + \left[\frac{\lambda^2}{a} Y_m''(\lambda a) - \left(\frac{\lambda}{a} + \frac{m^2\lambda}{a^2}\right) Y_m'(\lambda a) + \frac{2m^2}{\lambda^3} Y(\lambda a)\right]$$

$$\beta_m^J(\lambda a) = \lambda^3 J_m'''(\lambda a) + \left[\frac{\lambda^2}{a} J_m''(\lambda a) - \left(\frac{\lambda}{a} + \frac{m^2\lambda}{a^2}\right) J_m'(\lambda a) + \frac{2m^2}{\lambda^3} J(\lambda a)\right]$$

$$\beta_m^K(\lambda a) = \lambda^3 K_m'''(\lambda a) + \left[\frac{\lambda^2}{a} K_m''(\lambda a) - \left(\frac{\lambda}{a} + \frac{m^2\lambda}{a^2}\right) K_m'(\lambda a) + \frac{2m^2}{\lambda^3} K(\lambda a)\right]$$

$$\beta_m^I(\lambda a) = \lambda^3 I_m'''(\lambda a) + \left[\frac{\lambda^2}{a} I_m''(\lambda a) - \left(\frac{\lambda}{a} + \frac{m^2\lambda}{a^2}\right) I_m'(\lambda a) + \frac{2m^2}{\lambda^3} I(\lambda a)\right]$$

$$\gamma_m^Y(\lambda a) = m^2\left[\frac{1}{a^2} Y_m(\lambda a) - \frac{1}{a}\lambda Y_m'(\lambda a)\right]$$

$$\gamma_m^J(\lambda a) = m^2\left[\frac{1}{a^2} J_m(\lambda a) - \frac{1}{a}\lambda J_m'(\lambda a)\right]$$

$$\gamma_m^K(\lambda a) = m^2\left[\frac{1}{a^2} K_m(\lambda a) - \frac{1}{a}\lambda K_m'(\lambda a)\right]$$

$$\gamma_m^I(\lambda a) = m^2\left[\frac{1}{a^2} I_m(\lambda a) - \frac{1}{a}\lambda I_m'(\lambda a)\right]$$

$$\alpha_m^Y(\lambda b) = \lambda^2 Y_m''(\lambda b) + v\left[\frac{1}{b}\lambda Y_m'(\lambda b) - \left(\frac{m}{b}\right)^2 Y_m(\lambda b)\right]$$

$$\alpha_m^J(\lambda b) = \lambda^2 J_m''(\lambda b) + v\left[\frac{1}{b}\lambda J_m'(\lambda b) - \left(\frac{m}{b}\right)^2 J_m(\lambda b)\right]$$

$$\alpha_m^K(\lambda b) = \lambda^2 K_m''(\lambda b) + v\left[\frac{1}{b}\lambda K_m'(\lambda b) - \left(\frac{m}{b}\right)^2 K_m(\lambda b)\right]$$

$$\alpha_m^I(\lambda b) = \lambda^2 I_m''(\lambda b) + v\left[\frac{1}{b}\lambda I_m'(\lambda b) - \left(\frac{m}{b}\right)^2 I_m(\lambda b)\right]$$

$$\beta_m^Y(\lambda b) = \lambda^3 Y_m'''(\lambda b) + \left[\frac{\lambda^2}{b} Y_m''(\lambda b) - \left(\frac{\lambda}{b} + \frac{m^2\lambda}{b^2}\right) Y_m'(\lambda b) + \frac{2m^2}{\lambda^3} Y(\lambda b)\right]$$

$$\beta_m^J(\lambda b) = \lambda^3 J_m'''(\lambda b) + \left[\frac{\lambda^2}{b} J_m''(\lambda b) - \left(\frac{\lambda}{b} + \frac{m^2\lambda}{b^2}\right) J_m'(\lambda b) + \frac{2m^2}{\lambda^3} J(\lambda b)\right]$$

$$\beta_m^K(\lambda b) = \lambda^3 K_m'''(\lambda b) + \left[\frac{\lambda^2}{b} K_m''(\lambda b) - \left(\frac{\lambda}{b} + \frac{m^2\lambda}{b^2}\right) K_m'(\lambda b) + \frac{2m^2}{\lambda^3} K(\lambda b)\right]$$

$$\beta_m^I(\lambda b) = \lambda^3 I_m'''(\lambda b) + \left[\frac{\lambda^2}{b} I_m''(\lambda b) - \left(\frac{\lambda}{b} + \frac{m^2\lambda}{b^2}\right) I_m'(\lambda b) + \frac{2m^2}{\lambda^3} I(\lambda b)\right]$$

$$\gamma_m^Y(\lambda b) = m^2\left[\frac{1}{b^2} Y_m(\lambda b) - \frac{1}{b}\lambda Y_m'(\lambda b)\right]$$

$$\gamma_m^J(\lambda b) = m^2\left[\frac{1}{b^2} J_m(\lambda b) - \frac{1}{b}\lambda J_m'(\lambda b)\right]$$

$$\gamma_m^K(\lambda b) = m^2\left[\frac{1}{b^2} K_m(\lambda b) - \frac{1}{b}\lambda K_m'(\lambda b)\right]$$

$$\gamma_m^I(\lambda b) = m^2\left[\frac{1}{b^2} I_m(\lambda b) - \frac{1}{b}\lambda I_m'(\lambda b)\right]$$

ACKNOWLEDGEMENTS

Financial support from the National Science Council under Grant No. NSC-92-2211-E-019-013 for National Taiwan Ocean University is gratefully acknowledged. The Wu Ta-You award to the first author is also highly appreciated.

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